The Power Dissipation Method and Kinematic Reducibility of Multiple Model Robotic Systems

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***Abstract*—This paper develops a formal connection between the Power Dissipation Method and Lagrangian mechanics, with specific application to robotic systems. Such a connection is necessary for understanding how some of the successes in motion planning and stabilization for smooth kinematic robotic systems can be extended to systems with frictional interactions and overconstrained systems. We establish this connection using the idea of a multiple model system, and then show that multiple model systems arise naturally in a number of instances, including those arising in cases traditionally addressed using the Power Dissipation Method. We then give necessary and sufficient conditions for a dynamic multiple model systems to be reducible to a kinematic multiple model system. We are particularly motivated by mechanical systems undergoing multiple intermittent frictional contacts, such as distributed manipulators, overconstrained wheeled vehicles, and objects that are manipulated by grasping or pushing. Examples illustrate how these results can provide insight into the analysis and control of physical systems.**

***Index Terms*—quasi-static analysis, dynamics, contact modeling, frictional contacts, kinematic reducibility, modeling for control.**

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***Abstract*—This paper develops a formal connection between the Power Dissipation Method and Lagrangian mechanics, with specific application to robotic systems. Such a connection is necessary for understanding how some of the successes in motion planning and stabilization for smooth kinematic robotic systems can be extended to systems with frictional interactions and overconstrained systems. We establish this connection using the idea of a multiple model system, and then show that multiple model systems arise naturally in a number of instances, including those arising in cases traditionally addressed using the Power Dissipation Method. We then give necessary and sufficient conditions for a dynamic multiple model systems to be reducible to a kinematic multiple model system. We are particularly motivated by mechanical systems undergoing multiple intermittent frictional contacts, such as distributed manipulators, overconstrained wheeled vehicles, and objects that are manipulated by grasping or pushing. Examples illustrate how these results can provide insight into the analysis and control of physical systems.**

I. Introduction

Many mechanical systems, though intrinsically second order in their governing dynamics, can be adequately described by first order equations of motion. That is, one can often propose a "quasi-static" or "kinematic" version of the governing equations of motion for the purposes of system analysis or control design. The benefits of this simplification are numerous: the dimension of the state space drops by half, the control inputs go from being force inputs to being velocity inputs (which are often more easily realized in practice), and the governing equations typically take a simpler form than the full dynamic model. Additionally, kinematic systems, although potentially nonlinear, do not typically involve drift terms. There is a greater quality and quantity of nonlinear control results available for driftless systems, as compared to systems with drift. See [3], [10], [16], [17], [22], [36], [41] for just a few examples.

This paper has several inter-related goals. One of the main technical goals of this paper is to determine the formal conditions under which such reductions can be achieved for *multiple model systems*. In multiple model systems (see Section IV) the system's governing equations switch between several possible models that describe the system's evolution. This paper presents necessary and sufficient conditions for a multiple model system to be kinematically reducible—i.e., the 2^{nd}-order dynamical models can be reduced to 1^{st}-order kinematic models of the form in Definition IV.1. The necessary and sufficient conditions for kinematic reducibility of smooth dynamical systems were first developed by Lewis [23]. One of this paper's contributions is the extension of kinematic reducibility theory to the multiple model case.

While our kinematic reducibility results can be applied to a large class of problems, we are particularly motivated by the multiple model systems that arise frequently in robotics practice. The multiple model framework has received an increasing amount of attention in the control community recently [4], [19], [20], [18], so there are many control results available for our use. Therefore, understanding the connection between problems in robotics and the multiple model framework will be productive. Examples of multiple model systems include robotic systems involving intermittent mechanical contacts, such as distributed manipulators, overconstrained wheeled vehicles, and objects that are manipulated by grasping or pushing (see Section X). A number of similar approaches have been proposed or used to create quasi-static models of such systems. Most representative of these is the Power Dissipation Method (PDM) (see Section V) introduced by Alexander and Maddocks [3] in the context of overconstrained wheeled vehicles. Peshkin also used similar ideas in the study of pushed objects [39]. Based on this method, one can develop first-order (or quasi-static) equations of motion for mechanical systems that undergo intermittent sliding contacts. We show in Section VII that solutions to the PDM are multiple model systems. We have used the PDM to model distributed manipulation systems that generate motion via frictional contacts [34], [31]. The resulting multiple model descriptions are very amenable to control analysis [33], [30],

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and the associated nonsmooth control laws worked well in practice. See [34] for details.

As a second goal of this paper, we address a key question: does the PDM produce models that are consistent with a complete dynamic (Lagrangian) analysis? The formalization of the PDM and the analysis of its relationship to Lagrangian analysis are the other main contributions of this work. Formally, in Section VIII we show that every solution to the power dissipation method is precisely a reduction of a solution to the Lagrangian formulation. Moreover, this is true for *all* solutions, which is important, as solutions are not unique in either the power dissipation method nor are they unique in the Lagrangian formulation (when nonsmooth interactions such as impacts and friction are taken into consideration).

The paper is organized as follows. To motivate our results, we first examine some examples of mechanisms that naturally involve stick/slip phenomenon in Section II. Then we briefly review the classical Lagrangian approach in Section III before covering the basic ideas of the multiple model formalism in Section IV. We then specifically address an example in Section VI using these ideas. In Section VII we cover characteristics of the power dissipation method and we then move on to reduction theory for multiple model systems in Section VIII. Section IX relates solutions to the power dissipation method to solutions to the Lagrangian analysis. We end in Section X with a detailed look at several examples where we have found our analysis practically useful.

II. Examples

To show the potential breadth of applications for our results, we summarize here four typical robotic and physical systems to which our theory applies (Fig. 1): a wheeled bicycle, the Rocky 7 prototype of the NASA Mars rover family, a distributed manipulation system whose function is to manipulate a planar object via roll-slide contacts, and a multi-fingered robotic hand. All of these systems are characterized by complex mechanical interactions involving contact mechanics and slip. More specifically, all of these systems can be modeled and analyzed using the multiple-model framework developed in this paper.

Consider the bicycle of Fig. 1(a) . For simplicity, we assume that the bicycle is constrained to move along a line, and that both wheels are actuated. (We will repeatedly return to this example, as it exhibits many of the features that are relevant to our discussions). Applying the exact same torque to both wheels is very difficult task, and thus this bicycle would typically experience small amounts of slipping in practice. More interestingly, this slipping is likely to change over time due to variability in contact friction characteristics, leading to a multiple model, or hybrid, mechanical system. The multiple model methodology introduced in this paper and companion papers is well suited to analyze such systems.

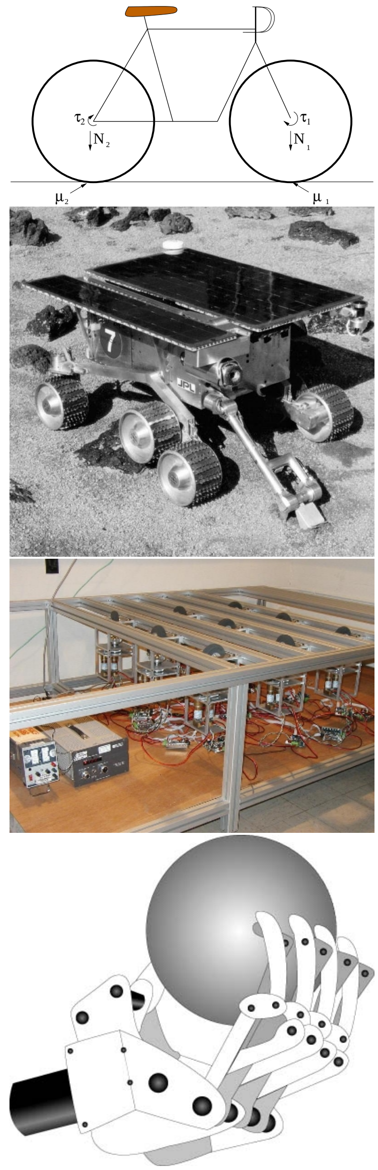

Fig. 1. Here are a) a bicycle with both wheels driven, b) the Mars rover Rocky 7 Sojourner prototype, c) a distributed manipulation test bed developed at Caltech (see description below), and d) a hand capable of grasping objects

The NASA Mars rover family members have six independently driven wheels as well as two wheels independently steered. As discussed in [29] and reviewed in Section X, because this vehicle's suspension is kinematically overconstrained, some of these wheels are always slipping, and it can be difficult to predict which wheels slip at any given moment. There is already an extensive literature on wheeled vehicles, establishing controllability based on a Lie Algebra Rank Condition (LARC) [21], [35], stability based on center manifold theory [41] and hybrid systems theory [18], motion planning based on Voronoi diagrams [9], and rapidly exploring random trees [10]. However, all of these methods assume that the vehicle motions are governed by smooth, kinematic equations of motion. Because of the inherent and unpredictable switches in slipping, the governing dynamics are not smooth. Nevertheless, the methods developed in this paper show that such vehicles are still kinematic systems, albeit nonsmooth ones. Moreover, in related work, we have made progress on extending classical nonlinear control concepts, such as the LARC, to the domain of multiple model systems [32]. We will discuss this more in Section X-B.

Distributed manipulation has received recent attention in the robotics community [6], [26]. Fig. 1(c) shows a distributed manipulation test-bed developed by the authors in which nine actuated wheels can be used to manipulate planar objects set upon the manipulation surface. All of these wheels can be independently driven and steered, giving the system 18 control inputs, with only the position and orientation of the manipulated object as the output. Hence, this system is massively over-actuated. The idea of many actuated devices interacting with an object to achieve some desired manipulation goal is appealing partially because of its scalability and the possibility of using many inexpensive actuators rather than a few expensive ones. Moreover, micro-electromechanical system (MEMS) fabrication technologies potentially enable distributed manipulation to be a leading candidate for micro-manipulation. We have shown in prior work how distributed manipulators that employ frictional contacts fall into the multiple-model domain [34]. The multiple model kinematic reducibility theory developed in this paper provides a simple but rigorous framework for the design of stabilizing control laws that take into account the non-smooth effects of friction. We have used kinematic reductions both to show the potential shortcomings of control laws based on smooth idealizations and to explicitly compute stabilizing control laws that work well experimentally (see [34]).

Grasping and locomotion continue to be active areas of robotics research. Current methods often use kinematic models [16], [17] to represent the system dynamics, yet grasping implicitly contains many of the previously mentioned difficulties. In particular, although stick/slip phenomena occur in a grasping problem, there are not very convincing ways to show that the kinematic methods typically used for grasping are robust with respect to the variation in stick and slip states for a given contact. The analytical methods presented here create a method for analyzing these difficulties without resorting to dynamic, second order analysis.

In Section X we will revisit these examples in order to show how the kinematic reduction theory of this paper can provide simplification or insight.

III. Background: Lagrangian models with Frictional contacts

This work has been largely motivated by the problem of modeling and controlling mechanical systems which experience multiple, possibly intermittent, contacts that involve friction, particularly Coulomb friction. Clearly, the contacts place constraints on the system's evolving motions. Constrained mechanical systems can be modeled using conventional Lagrangian mechanics through the use of Lagrange multipliers. Consider a generic mechanical system with up to n frictional contacts between rigid body surfaces, where the contacts can be intermittently slide or stick. Such a system admits up to 2^n possible contact states which represent all possible permutations of sliding and sticking. Let $L(q,\dot{q})$ denote the system's Lagrangian (kinetic minus potential energy), where $q \in Q$ denotes the configuration of the mechanical system, Q is its the configuration space, which is assumed to be an n-dimensional manifold. If the i^{th} physical contact does not slip, the contact imposes a nonholonomic constraint on the mechanical system's motion. This constraint can be expressed in the form $\omega_i(q)\dot{q} = 0$. If the i^{th} contact slips, the Coulomb friction law states that the tangential reaction force at that contact is $F_i^T = -\frac{v_i}{||v_i||}\mu_i F_i^N$, where μ_i, F_i^N, and v_i are respectively the Coulomb friction coefficient, normal force to the contacting surface, and slipping velocity of the contact at the i^{th} contact. Hence, the mechanical system's overall equations of motion can described by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \sum_{i\in\mathcal{S}} F_i^T + \sum_{j\notin\mathcal{S}} \lambda_j \omega_j^T(q) = T \quad (1)$$

where $\mathcal{S}$ is the *slipping contact set*, the $\{\lambda_j\}$ are undetermined Lagrange multipliers, and T are the generalized applied forces. That is, $k \in \mathcal{S}$ if the k^{th} contact is slipping. If the k^{th} contact is not slipping, λ_k corresponds to the tangential reaction force that is needed to maintain the no-slip constraint at the k^{th} contact. We generally assume in this work that the contact normal forces, $\{F_i^N\}$ are known. If this is not the case, then additional Lagrange multipliers may typically be added to solve for these normal forces. Note that this description involves a choice of coordinates. The equivalent, coordinate independent, representation is the formalism in which we address these problems, and is briefly reviewed in the Appendix.

There are two primary practical problems with the Lagrangian modeling approach. First, one must solve for the Lagrange multipliers—a tedious task that often leads to complex equations. Second, an additional (and often sensitive) analysis is necessary to determine which contacts are slipping at any given instant. Consequently, the practical need to analyze such systems in a tractable way motivates the use of quasi-static or kinematic approximations, and in particular the Power Dissipation Method that is reviewed in the Section V. A natural question arises when using quasistatic analysis: what is the relationship between the equations of motion predicted by quasi-static analysis and those generated by Lagrangian analysis? Moreover, can the quasistatic equations properly predict the motions of the true system? The next section briefly reviews the concept of a multiple-model system, which is the appropriate mathematical setting for this question in the case of intermittent frictional contacts. We describe a method for finding quasistatic equations of motion in Section V and we answer these questions in Section IX.

IV. Background: Multiple Model Systems

We use the formalism of multiple model systems to address kinematic reducibility of systems involving frictional and intermittent contact.

Definition IV.1: A control system Σ evolving on a smooth n-dimensional manifold, Q, is said to be a *multiple model driftless affine system (MMDA)* if it can be expressed in the form

$$\Sigma : \qquad \dot{q} = f_1(q)u_1 + f_2(q)u_2 + \cdots + f_m(q)u_m \quad (2)$$

where $q \in Q$. For any q and t, the vector field f_i assumes a value in a finite set of vector fields: $f_i \in \{g_{\alpha_i} | \alpha_i \in I_i\}$, with I_i an index set. The vector fields g_{α_i} are assumed to be analytic in (q,t) for all α_i, and the controls $u_i \in \mathbb{R}$ are piecewise constant and bounded for all i. Moreover, letting σ_i denote the "switching signals" associated with f_i

$$\begin{array}{rccc} \sigma_i : & Q \times \mathbb{R} & \longrightarrow & \mathbb{N} \\ & (q,t) & \longrightarrow & \alpha_i \end{array}$$

the σ_i are measurable in (q,t).

Definition IV.1 implies that the control vector fields may change, or switch, among a finite collection of vector fields, each representing a single smooth model in a set of models $\mathcal{P}$. An example of such a system is a vehicle whose wheels can potentially skid. The system's governing dynamics will vary when the wheels slip or do not slip. Such systems are intimately related to multiple model systems such as studied in [18]. However, we should emphasize that the "switching" is *not* like the switching phenomena found in [8], [25], [12], [44], or as typically studied in the hybrid control systems literature (e.g., [38], [5]). In these studies, the switching phenomena is part of a control strategy to be implemented in the controller. In our case, the switching is induced by environmental factors, such as variations in the contact state between rigid bodies. Since the phenomena which govern the switching behavior may not be precisely characterized, we make no assumptions about the nature of the switching functions, except that they are measurable. A long term goal of our work is to develop systematic methods for analyzing control systems with the type of hybrid (and therefore nonsmooth) structure seen in Definition IV.1.

To distinguish between the overall control system and the smooth control systems that comprise it, we define the *individual control systems* to be the smooth control systems making up the multiple model system, comprising of $\dot{q} = g_1u_1 + \cdots + g_ku_k \cdots + g_nu_n$ for $g_k = g_{\alpha_i}$ for some α_i. A system will be termed a *multiple model affine* system if it has the form $\dot{q} = f_0(q) + f_1(q)u_1 + f_2(q)u_2 + \cdots + f_m(q)u_m$, where the vector field $f_0(q)$ (or "drift term") is also selected from a set of analytic vector fields g_{σ_0}.

V. Overview of The Power Dissipation Methodology

This section reviews the basic concept behind the Power Dissipation Method (PDM), which we will formalize in Section VII. Let q again denote a system configuration. The relative motions between moving objects at a point contact can be written in the form $\omega(q)\dot{q}$. If $\omega(q)\dot{q} = 0$, then the contact point is not slipping, while if $\omega(q)\dot{q} \neq 0$, then $\omega(q)\dot{q}$ describes the contact point's slipping velocity. The *power dissipation function* measures the object's total frictional energy dissipation due to contact slippage.

Definition V.1: Consider a mechanical system, $\mathcal{S}$ (which consists of a single rigid body or a set of rigid bodies) that maintains n frictional contacts, where some or all of the contacts may be slipping. The *Dissipation* or *Friction Functional* for n-contact states that are governed by Coulomb friction is defined to be

$$\mathcal{D} = \sum_{i=1}^{n} \mu_i N_i \mid \omega_i(q)\dot{q} \mid \quad (3)$$

where $\omega_i(q)\dot{q}$ describes the relative slipping velocity, μ_i is the Coulomb friction coefficient, and F_i^N is the normal force at the i^{th} contact.

The form of this function reflects the Coulomb friction model, but it can easily be extended to different friction models (see [37]) by replacing the linear term $\mu_i N_i$ with a more general state-dependent function, $h_i(q)$. Every slipping contact dissipates energy. Based on this observation, Alexander and Maddocks proposed the following axiom:

Power Dissipation Principle: A system's motion at any given instant is the one that minimizes $\mathcal{D}$ (Eq. 3) with respect to $\dot{q}$.

The *power dissipation method* is built upon this axiom. That is, the first order equations of motion generated by the system are precisely the ones that minimize the dissipation.

Remark V.1: Some insight into the relationship between the motions predicted by the PDM and those given by the Lagrangian approach can be seen in the following example. Consider a particle constrained to move on a surface, with friction between the particle and the surface. Lagrangian analysis suggests that there are two possible contact states–one slipping and one not slipping. The PDM predicts that the particle will not slip. Hence, it misses some of the contact states predicted by the Lagrangian framework. However, the non-slip motions that it does predict are consistent with a Lagrangian analysis.

For overconstrained systems with control inputs, the PDM leads to more interesting and useful results. When a configuration q can be decomposed into two components $q = (s, r)$ (where we refer to s as the group variable and r as the shape variable), Eq. (3) implies that the PDM will predict $\dot{s}$ given $\dot{r}$. In most cases of interest, the variable $\dot{r}$ corresponds to the control inputs, while the variables $\dot{s}$ corresponds the system motion of interest.

VI. Example: A Two-Wheeled Bicycle

Consider the planar bicycle (Fig. 1(a)) which is constrained to move along a line. We will revisit this example using the PDM formalism, but for now we treat it in the Lagrangian framework. Let $q = [x, \phi_1, \phi_2]^T$, where ϕ_1 is the front wheel angle, ϕ_2 is the rear wheel angle, and x denotes the bicycle's translation along the line. The downward normal force on each wheel depend upon the bicycle's weight distribution, which is assumed to be known. Assume that each wheel is actuated, with torques τ_1 and τ_2, and that each wheels may possibly slip.

Using Eq. (1) and solving for the Lagrange multipliers, there are four different governing equations of motion (see Table I), each corresponding to a different type of contact state. The analysis based on Lagrangian mechanics suggests that there are *four* possible contact states, corresponding to Eq. (A) where neither wheel slips, Eq. (B) where the front wheel slips, Eq. (C) where the rear wheel slips, and Eq. (D) where both wheels slip.

$$\ddot{q} = \begin{bmatrix} \frac{R}{2J+mR^2} \\ \frac{1}{2J+mR^2} \\ \frac{1}{2J+mR^2} \end{bmatrix} \tau_1 + \begin{bmatrix} \frac{R}{2J+mR^2} \\ \frac{1}{2J+mR^2} \\ \frac{1}{2J+mR^2} \end{bmatrix} \tau_2 \qquad \text{(A)}$$

$$\ddot{q} = \begin{bmatrix} \frac{F_1^R}{J+mR^2} \\ \frac{-RF_1^R}{J} \\ \frac{RF_1^R}{J+mR^2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \\ 0 \end{bmatrix} \tau_1 + \begin{bmatrix} \frac{R}{J+mR^2} \\ 0 \\ \frac{1}{J+mR^2} \end{bmatrix} \tau_2 \qquad \text{(B)}$$

$$\ddot{q} = \begin{bmatrix} \frac{F_2^R}{J+mR^2} \\ \frac{RF_2^R}{J+mR^2} \\ \frac{-RF_2^R}{J} \end{bmatrix} + \begin{bmatrix} \frac{R}{J+mR^2} \\ \frac{1}{J+mR^2} \\ 0 \end{bmatrix} \tau_1 + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J} \end{bmatrix} \tau_2 \qquad \text{(C)}$$

$$\ddot{q} = \begin{bmatrix} \frac{F_1^R+F_2^R}{m} \\ \frac{-F_1^R R}{J} \\ \frac{-F_2^R R}{J} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \\ 0 \end{bmatrix} \tau_1 + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J} \end{bmatrix} \tau_2 \qquad \text{(D)}$$

TABLE I

THE LAGRANGIAN DYNAMICS OF THE PLANAR BICYCLE IN THE FOUR POSSIBLE CONTACT STATES. J IS A WHEEL'S MOMENT OF INERTIA ABOUT ITS ROTATIONAL AXIS, m IS TOTAL BICYCLE MASS, AND R IS THE WHEEL RADIUS.

When the i^{th} wheel slips, the tangential reaction force at the i^{th} contact point is governed by the Coulomb friction law: $F_i^T = -\frac{\dot{x}-R\dot{\phi}_i}{\|\dot{x}-R\dot{\phi}_i\|}\mu_i F_i^N$, where μ_i is the Coulomb friction coefficient, and F_i^N is the normal force bearing down upon the i^{th} wheel contact. When the i^{th} wheel does not slip, the tangential reaction force is given by the Lagrange multiplier λ_i. The Coulomb friction model implies that the boundary between slipping and nonslipping states occurs at some value of the Lagrange multiplier, denoted by λ^{nom}. When $\lambda_i > \lambda_i^{nom}$, the i^{th} contact slips. Consequently, the λ space is divided into regions corresponding to different contact slipping states. The problem of contact state determination arises from the inherently complicated dependency of λ on the current state. For the planar bicycle model, the Lagrange multipliers assume the following values when model (A) holds:

$$\lambda_1 = \frac{J(\tau_1 - \tau_2) - R^2 m\tau_1}{R(R^2 m + 2J)} \qquad \lambda_2 = \frac{J(\tau_2 - \tau_1) - R^2 m\tau_2}{R(R^2 m + 2J)}.$$

Under the Coulomb friction model, the critical value of λ for this example takes the value $\lambda_{nom} = \mu_i F_i^N$. However, depending on the friction model λ_{nom} will take different values. This fact implies that the boundary of these regions is both terrain dependent and sensitive to the details of the friction model. One of the purposes of this paper is to provide a modeling foundation for control strategies that are not sensitive to the friction model, such as those we employ in [34].

Alexander and Maddocks showed that $\mathcal{D}$ is convex as a function of $\dot{q}$; therefore its local minima are global minima [3]. Note that the minimum of $\mathcal{D}$ must occur at a nondifferentiable point of $\mathcal{D}$, since the function is monotone everywhere else. By direct comparison of the two nondifferentiable states, which correspond to one of the wheels not slipping, the minimum is associated with whichever wheel is associated with a lower value of μN. Consequently, the zero level set of the function

$$\Psi(q) = \mu_1 N_1 - \mu_2 N_2$$

determines the contact state of the bicycle. This determination is nonunique when $\mu_1 N_1 = \mu_2 N_2$. (This is also true for the Lagrangian system.) This model has only two states, making it simpler to analyze than the Lagrangian model. Additionally, the governing equations take the simplified form:

$$\dot{x} = Ru_i \tag{4}$$

where i indexes the wheel not slipping and the u_i are velocity inputs.

To compare the PDM method to Lagrangian analysis, consider the bicycle example with torque inputs on both the front wheel W_1 and the back wheel W_2. The PDM predicts two different contact states corresponding to either the front or rear wheel slipping. In comparison, Lagrangian analysis predicts four possible contact states. Eqs. (A) and (D) in Table I both imply that the inertial terms dominate the system's dynamics, thereby violating the "quasi-static" assumption. Eq. (D) implies that the bicycle is skidding out of control. The physical conditions corresponding to Eq. (A) are unlikely to be found in an actual system, as they imply that both contacts must be driven at *exactly* compatible speeds. Moreover, this contact state will be predicted by the PDM so long as the two wheels are driven at exactly the same speed—we will see later these conditions can be interpreted as a special degenerate case. This leaves the second two contact states represented by Eqs. (B) and (C), which are the same as those found in Eq. (4) using the power dissipation model. This is an indication of how the quasi-static assumption helps to simplify our problem, while yielding results that are consistent with the contact state analysis of the Lagrangian. With the additional analyses introduced below, we can investigate the relationship between the motions predicted by the Lagrangian method and the PDM in comparable contact states.

VII. Characteristics of the PDM

In this section we formalize the Power Dissipation Method and show that the PDM generically gives rise to *multiple model driftless affine* systems, as described in IV.1.

Before proceeding, let us recall a few facts that were already established by Alexander and Maddocks [3]. They showed that the dissipation function of Eq. 3 is convex, so that its local minima are also its global minima, should they exist. They also show that if such a minimum exists, it must exist at a point of nondifferentiability of $\mathcal{D}$ due to the piecewise continuity of $\mathcal{D}$.

Let $\Omega = \{\omega_1, \cdots, \omega_m\}$ and denote the *constraint 1-forms*. Furthermore, let $\mathcal{Q} = \{\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_r\}$ consist of the $\binom{n}{n-m}$ velocities that have the property that $\dot{q}_k$ is a kinematic solution to a non-overconstrained subset $\Omega' \subset \Omega$ consisting of $n-m$ constraints, i.e.,

$$\Omega' \dot{q}_k = \begin{bmatrix} \omega_{k_1} \\ \vdots \\ \omega_{k_{m-n}} \end{bmatrix} \dot{q}_k = 0.$$

That is, at all points in $\mathcal{Q}$ the derivatives of $\mathcal{D}$ are nonsmooth. From Alexander and Maddocks, we have only a finite number of points to check in order to find the minima of $\mathcal{D}$. It is straightforward to show that these minima must *at least* occur at points in $\mathcal{Q}$. See, for instance, Clarke [11]. Reorder $\mathcal{Q}$ so that $\mathcal{D}(\dot{q}_1) \le \mathcal{D}(\dot{q}_2) \le \cdots \le \mathcal{D}(\dot{q}_r)$. Although $\mathcal{Q}$ is associated with at least one of the minima achieved by $\mathcal{D}$, it does not necessarily contain all of them. In fact, if more than one element of $\mathcal{Q}$ is a minimum, then every element of the convex hull of these minima are also minima. Hence, if there is more than one solution, there are an infinite number of solutions.

Proposition VII.1: If $\dot{q}_1$ and $\dot{q}_2$ both minimize the dissipation functional found in Definition V.1, then so does $co\{\dot{q}_1, \dot{q}_2\}$.

Proof: Assume $\mathcal{D}(\dot{q}_1) = \mathcal{D}(\dot{q}_2) = a$ and $\delta \in [0,1]$. Then

$$\mathcal{D}(q)\left(\delta\dot{q}_1 + (1-\delta)\dot{q}_2\right) = \sum_{i=1}^{n} \mu_i F_i^N \left|\omega_i\left(\delta\dot{q}_1 + (1-\delta)\dot{q}_2\right)\right|$$

$$\le \delta \sum_{i=1}^{n} \mu_i F_i^N \left|\omega_i\left(\dot{q}_1\right)\right| + (1-\delta) \sum_{i=1}^{n} \mu_i F_i^N \left|\omega_i\left(\dot{q}_2\right)\right| = a$$

Assume that $\mathcal{D}$ is strictly less than $\mathcal{D}(\dot{q}_1)$ somewhere in $co\{\dot{q}_1, \dot{q}_2\}$. Then $\exists\, \delta'$ such that $\mathcal{D}(\delta'\dot{q}_1 + (1-\delta')\dot{q}_2)$ is at a minimum by an extension of Rolle's Theorem for the real line [15]. Then $\dot{q}' = \delta'\dot{q} + (1-\delta')\dot{q}$ is at a point where $\mathcal{D}$ is nonsmooth in all its directional derivatives [3] (because $\mathcal{D}$ is monotone elsewhere). This implies that $\dot{q}' \in \mathcal{Q}$ and that $\mathcal{D}(\dot{q}') < \mathcal{D}(\dot{q}_1)$, thus violating our assumption that $\mathcal{D}(\dot{q}_1)$ is a minimum of $\mathcal{D}$. Therefore $\mathcal{D}(q)\left(\delta\dot{q}_1 + (1-\delta)\dot{q}_2\right) = a\ \forall \delta \in [0,1]$. The proof for higher numbers of $\dot{q}_i$ having equal dissipation is by induction on this argument. ■

This result formalizes the intuition that if the power dissipated is equal for two velocities $\dot{q}_i$, then all possible trajectories whose velocity lies in the convex hull of the $\dot{q}_i$ will satisfy the minimum also. That is, in the nongeneric case when $\mathcal{D}$ does not have a unique minimum, we can still bound the object's motion. (We will see later that these solutions correspond exactly to kinematic solutions of the Lagrangian dynamics.) Now $co\{\dot{q}_i, i \in J\}$ is a set of points on which $\mathcal{D}$ is nondifferentiable, just not in all directions. It therefore still meets the criterion to be a minimum [3]. Let us consider the extent to which the function $\mathcal{D}$ having a unique minimum is generic. We denote the function space of the coefficient of friction by Ξ, the function space of normal forces by $\mathcal{N}$.

Proposition VII.2: Assume $\mathcal{D} : (\mathcal{U}, \Xi, \mathcal{N}, TQ) \to \mathbb{R}$ is of the form in Definition V.1 and that the μ is measurable in x and t. Then the dissipation functional $\mathcal{D}$ has a unique minimum almost always (i.e., except on a set of measure zero[1] relative to the space $(\mathcal{U}, \Xi, \mathcal{N}, TQ)$)

Proof: **Case 1:** If $\dot{q}_1$ is a unique minimum in $\mathcal{Q}$, then it is the unique global minimum since Alexander and Maddocks showed that the minimum must occur in $\mathcal{Q}$.
Case 2: If $\exists\ \dot{q}_1$ and $\dot{q}_2$ such that both are minima, then by Proposition VII.1, we know that $co\{\dot{q}_1, \dot{q}_2\}$ also minimizes the $\mathcal{D}$. However, this situation can only occur when the parameters $(u_i, N_j, \mu_k) \in \mathcal{U} \times \mathcal{N} \times \Xi$ are chosen to satisfy the constraint $\mathcal{D}(\dot{q}_1) = \cdots = \mathcal{D}(\dot{q}_n)$. This implies that the constraint is only satisfied on a set of measure 0 in the space $\mathcal{U} \times \mathcal{N} \times \Xi$. ■

That is, the PDM will almost always lead to a unique set of governing equations. The reader should note that the proof of Proposition VII.2 is only useful if we have already found $\mathcal{Q}$, and moreover for a high number of states it may be computationally expensive to find the minimum of $\mathcal{Q}$.[2] Also note that in the non-overconstrained case of $n - m$ constraints, the dissipation method leads to the classical kinematic solution in the sense of the Appendix. Proposition VII.2 allows us to now state what we mean by the dissipation functional leading generically to an MMDA system. A direct consequence of Proposition VII.2 is the following Corollary.

Corollary VII.3: The multivalued map $\mathcal{F} : TQ \to TQ$ implicitly defined by $\mathcal{D}(\dot{q}) = min(\mathcal{D})$ is single valued almost everywhere.

Corollary VII.3 implies that we can generically expect the power dissipation method to lead to a unique and well defined set of first-order governing equations—it will almost never lead to an indeterminate system. This makes rigorous the comment made in [3] referring to the physical expectation of continually switching back and forth between the dominance of one wheel or another, rather than staying in an indeterminate state. See [13] for a discussion of implicitly defined multivalued maps. Corollary VII.3 additionally establishes a relationship between solutions that minimize $\mathcal{D}$ and MMDA systems. Moreover, we will see that the contact states predicted by the PDM are $(\mathcal{U},\overline{\mathcal{U}})$ reductions of a class of mechanical control systems on TQ.

Corollary VII.3 also implies that multiple model systems are a natural result of frictional interactions. Consequently, multiple model modeling and control techniques should be developed for systems involving frictional contact. In Section IX we will explore more formally the relationship between solutions to the PDM and solutions to the Lagrangian dynamics.

[1] Intuitively, sets of measure 0 can be as sparse as disjoint points in Q or as replete as a submanifold of Q. For example, consider a vehicle moving on smooth terrain. In its ambient Euclidean space, a vehicle is always constrained to a set of measure 0, yet that set is precisely where the interesting dynamics occur. On the other hand, sets of measure 0 can represent arbitrary algebraic relationships between parameters and the state space. Unless there is some reason to believe that these relationships are necessarily satisfied, we can feel physically motivated in asserting they will not occur in practice. This is the case that we are considering, and therefore we feel that the ensuing results do imply the genericity we assert. Nevertheless, whether or not these sets are important in the analysis is a *physical assumption*, not a mathematical result. For a reference on measure theory, see [2].

[2] This problem, thus stated, bears more than a passing resemblance to the simplex method found in LP theory and techniques from that theory can be applied to the problem of finding the minimum of the function $\mathcal{D}$ in the presence of high numbers of contact states.

VIII. KINEMATIC REDUCIBILITY FOR MULTIPLE MODEL SYSTEMS

This section introduces the formal tools and results required to relate solutions arising from the power dissipation method to solutions arising from the full Lagrangian analysis. A rigorous understanding of the PDM's properties and its relationship to conventional Lagrangian mechanical analysis has heretofore been missing. We structure our analysis of this issue in two steps. In the previous section we developed a more formal mathematical framework for the PDM. In particular, we showed that the PDM leads generically to multiple model systems. This section introduces kinematic reducibility theory for multiple model systems. We then use our multimodel reduction theory to formally study the relationship between the properties of the PDM solutions and those of the associated Lagrangian models (in Section IX.2).

A. Review of Kinematic Reducibility for Smooth Systems

We briefly review the relevant notions of kinematic reduction here, without going into details of the underlying formalism. For some of these details, refer to the Appendix and to [23]. First we start with what we mean by a solution to a control system. In the following, Q is the configuration space and TQ is its tangent bundle. Moreover, if $\{X_i\}$ are kinematic vector fields and $\{Y_j\}$ are dynamic vector fields (see the Appendix for notational details), we let the *distributions* D_{kin} and D_{dyn} be defined by $D_{kin} = span\{X_i\}$ and $D_{dyn} = span\{Y_j\}$.

Definition VIII.1: Let Σ_s be a smooth control system $\dot{q} = f(q, u)$ on Q and let $u \in U \subseteq \mathbb{R}^m$. A $(\mathcal{U}, \mathcal{T})$-*solution* to Σ_s is a pair (c, u), where $u : [0, T] \to U$ and $c : [0, T] \to Q$ satisfy $c'(t) = f(c(t), u(t))$.
We now can define what it means for a mechanical system of the form in Eq. (20) to be $(\mathcal{U},\overline{\mathcal{U}})$ reducible to Eq. (21). Let τ_Q

$$\begin{array}{rccc} \tau_Q : & TQ & \to & Q \\ & (v_q, q) & \to & q \end{array}$$

denote the tangent bundle projection.

Definition VIII.2: Let ∇ be an affine connection on Q (see the Appendix), and let $\mathcal{U}$ and $\overline{\mathcal{U}}$ be two families of control functions. The system in Eq. (20) is $(\mathcal{U},\overline{\mathcal{U}})$-*reducible* to the system in Eq. (21) (also in the Appendix) if the following two conditions hold:

i) for each $(\mathcal{U},\mathcal{T})$-solution (η,u) of the dynamic Eq. (20) with initial conditions $\eta(0)$ in the distribution D_{kin}, there exists a $(\overline{\mathcal{U}},\mathcal{T})$-solution $(\gamma,\overline{u})$ of the kinematic Eq. (21) with the property that $\gamma=\tau_Q\circ\eta$;

ii) for each $(\overline{\mathcal{U}},\mathcal{T})$-solution $(\gamma,\overline{u})$ of the kinematic Eq. (21), there exists a $(\mathcal{U},\mathcal{T})$-solution (η,u) of the dynamic Eq. (20) with the property that $\eta(t)=\gamma'(t)$ for almost every $t\in[0,T]$.

Condition i) says that for every solution of a dynamic system there must exist a kinematic solution that is the projection of the dynamic system. In the case of a vehicle, this corresponds to requiring that for every *trajectory* of the vehicle there exists a corresponding *path* that can be obtained from kinematic considerations alone. Condition ii) says that every kinematic solution must be the integral of a dynamic solution. For a vehicle, this means that there must exist a dynamic solution for every feasible kinematic path. We should point out here that this is related to the classes of admissible inputs. Because kinematic inputs must be essentially integrals of dynamic inputs, they must be absolutely continuous if the dynamic inputs are measurable. Otherwise, infinite forces would be required (see [23]).

Let $\chi^\infty(D)$ denote those C^∞ vector fields taking values in a distribution D. The following theorem states the local test for Eq. (20) to be $(\mathcal{U},\overline{\mathcal{U}})$ reducible to Eq. (21).

Theorem VIII.1—Lewis [23]: Let ∇ be an affine connection, and let $Y_1,\ldots,Y_m$ and $X_1,\ldots,X_{\overline{m}}$ be vector fields on a manifold Q. The control system in Eq. (20) is $(\mathcal{U},\overline{\mathcal{U}})$ *reducible* to a system of the form in Eq. (21) if and only if the following two conditions hold:

i) $span_{\mathbb{R}}\{X_1(q),\ldots,X_m(q)\} = span_{\mathbb{R}}\{Y_1(q),\ldots,Y_{\overline{m}}(q)\}$ for each $q\in Q$ (in particular, $\overline{m}=m$)

ii) $\langle X:Y\rangle \in \chi^\infty(D_{dyn})$ for every $X,Y\in\chi^\infty(D_{dyn})$ where $\langle\cdot,\cdot\rangle$ is the symmetric product of vector fields, defined in the Appendix.

This theorem says that if the input distributions of both the kinematic system and the dynamic system are the same and the dynamic system is closed under symmetric products, then the system is kinematic.

B. Main Result on Reducibility of Multiple Model Systems

We now consider the problem of whether or not a dynamic multiple model system is kinematically reducible to an MMDA system. Lemma VIII.2 states that if switches in system dynamics are separated by a small amount of time (making the switching signal piecewise continuous), the resulting solution is also kinematically reducible.

Lemma VIII.2: Let Σ be a multiple model system whose switching signal σ is piecewise constant. Then, Σ is $(\mathcal{U},\overline{\mathcal{U}})$ reducible iff the individual model components $\Sigma_{\sigma_i,\cdots,\sigma_j}$ are all $(\mathcal{U},\overline{\mathcal{U}})$ reducible.

Proof: Since σ is piecewise constant, σ switches a countable number of times. Therefore, let the times when σ changes its value be denoted $\{t_1,t_2,\cdots,\}$ for i in some index set I. Then on the intervals (t_i,t_{i+1}), Σ is $(\mathcal{U},\overline{\mathcal{U}})$ reducible, making it $(\mathcal{U},\overline{\mathcal{U}})$ reducible almost always.[3] It therefore satisfies the requirements of Definition VIII.2. ■

We will use this lemma to prove Theorem VIII.4, which says that solutions to the differential inclusion defined by multiple model systems are kinematically reducible if and only if the individual models are kinematically reducible. Before proving that this is true, we will need the following result from Filippov [14].

Theorem VIII.3—Filippov [14]: Let $\mathbf{f}:Q\times\mathbb{R}\to TQ$ be a compact, set-valued map and let $\{\Phi_i\}$ be a sequence of solutions to the differential inclusion

$$\dot{q}\in\mathbf{f}(t,q) \tag{5}$$

such that $\lim_{i\to\infty}\Phi_i\to\Phi$. Then Φ is also a solution to Eq. (5).

Note that solutions to the differential inclusion $\mathbf{f}$ are in general not unique, meaning that there is often an infinite family of solutions. This theorem says that for a compact differential inclusion, a converging sequence of solutions converges to a solution. Theorem VIII.3 will be used several times in the proof of Theorem VIII.4. Roughly speaking, piecewise continuous $(\mathcal{U},\overline{\mathcal{U}})$ reducible solutions of the multiple model mechanical system can be used as approximations to flows of elements in $\mathbf{f}$, where $\mathbf{f}$ assumes the form of the right half side of Eq. (6). Theorem VIII.3 can then be used to show that their kinematic counterparts on TQ must also converge to an element of the differential inclusion defined on TQ. This brings us to our main result.

Theorem VIII.4: A multiple model system Σ is $(\mathcal{U},\overline{\mathcal{U}})$ reducible iff the individual dynamical models $\Sigma_{\sigma_i,\cdots,\sigma_j}$ are all $(\mathcal{U},\overline{\mathcal{U}})$ reducible.

Proof: First note that it is obviously necessary that all the individual models be $(\mathcal{U},\overline{\mathcal{U}})$-reducible in order for the resulting multiple model system to be reducible. Otherwise, a valid solution to a multiple model system is the smooth, non-reducible solution of one of the models in the set of models. To show sufficiency, we must show that when the individual models are $(\mathcal{U},\overline{\mathcal{U}})$ reducible,

[3] That is, it is reducible everywhere except for a set of measure 0.

the MMDA system satisfies parts *i)* and *ii)* of Definition VIII.2. We show this in two steps. The first step constructs kinematic solutions given dynamic ones, and the second step constructs dynamic solutions given kinematic ones.

(i) A multiple model mechanical system has the form (see the Appendix for notation)

$$^{G_l}\nabla_{c'(t)}c'(t) \in u^{\alpha}\ {}^{l}Y_{\alpha}(c(t)) \tag{6}$$

where $l \in \Lambda \subset \mathbb{N}$ is the index for a given model, G_l is the metric appropriate to that model, $^{G_l}\nabla$ is the affine connection associated with the metric G_l, and $^{l}Y_{\alpha}$ is the vector field representing the force input corresponding to u^{α} of the l^{th} model of the multiple model system. In coordinates, Eq. (6) is equivalent to

$$\ddot{q}^i + {}^{G_l}\Gamma^i_{jk}\dot{q}^j\dot{q}^k = u^{\alpha}\ {}^{l}Y^i_{\alpha}. \tag{7}$$

Set ${}^{l}\mathcal{Y}^i = -{}^{G_l}\Gamma^i_{jk}\dot{q}^j\dot{q}^k + u^{\alpha}\ {}^{l}Y^i_{\alpha}$ and $\mathbf{Y}^i = co\{{}^{l}\mathcal{Y}^i : l \in \Lambda\}$, with $co\{\cdot,\cdot\}$ denoting the convex hull. In [14] it was shown that solutions to a discontinuous system coincide with solutions of a differential inclusion of the convex hull of the discontinuous system. Applying this to our systems of interest, we see that solutions to a multiple model system coincide with solutions to the differential inclusion $\ddot{q}^i \in \mathbf{Y}^i$, or in vector notation:

$$\ddot{q} \in \mathbf{Y}. \tag{8}$$

Eq. (8) is a second order system on Q that we can easily rewrite as a first order system on TQ (see [23] for details of this procedure). Then, for a given solution $\Phi(t)$ of Eq. (8) rewritten as a first order system, we know that $\frac{d}{dt}\Phi \in \mathbf{Y}$. Therefore, we can choose a selection (an element) of $\mathbf{Y}$, denoted $s(\mathbf{Y}) \in \mathbf{Y}$, such that $\Phi^{s(\mathbf{Y})}$ locally approximates the flow Φ. Because $\mathbf{Y}$ is convex, we can rewrite a selection of $\mathbf{Y}$ as

$$s(\mathbf{Y}) = \delta_1\ {}^{1}\mathcal{Y} + \delta_2\ {}^{2}\mathcal{Y} + \cdots + \delta_m\ {}^{m}\mathcal{Y} \tag{9}$$

for any δ_j such that $\delta_j > 0$ and $\sum_j^m \delta_j = 1$. Let us denote the composition of a flow Φ with itself n times by Φ^n. That is, $\Phi^n(q) = \Phi\circ\Phi\circ\cdots\Phi\circ\Phi(q)$. In [28], it was shown that we can choose the following map to approximate (in the sense of pointwise convergence to a set) the flow of the selection $s(\mathbf{Y})$:

$$\Phi^{t,n}_{dyn}(q) \overset{def}{=} \left(\Phi^{\delta_1\ {}^{1}\mathcal{Y}\frac{t}{n}} \circ \Phi^{\delta_2\ {}^{2}\mathcal{Y}\frac{t}{n}} \circ \cdots \circ \Phi^{\delta_m\ {}^{m}\mathcal{Y}\frac{t}{n}}\right)^n(q) \tag{10}$$

Each of the component flows $\Phi^{\delta_m\ {}^{i}\mathcal{Y}\frac{t}{n}}$ contributing to the flow $\Phi^{t,n}_{dyn}(q)$ consists of a flow along a $(\mathcal{U},\overline{\mathcal{U}})$ reducible mechanical system. Moreover, $\Phi^{t,n}_{dyn}(q)$ is a solution of Eq. (8) on TQ which is absolutely continuous for every n. This is due to the fact that we assume that the switching is measurable and the forces are measurable and that the Lebesgue integral of measurable signals is absolutely continuous. Lastly, it converges to the flow of the selection $s(\mathbf{Y})$ as $n \to \infty$. That is, by applying Theorem VIII.3 to the Taylor expansion of $\Phi^{t,n}_{dyn}$, we locally get

$$\lim_{n\to\infty}\Phi^{t,n}_{dyn} = \Phi^{s(\mathbf{Y})}.$$

By assumption, we know that each segment $\Phi^{\delta_i\ {}^{i}\mathcal{Y}\frac{t}{n}}$ of $\Phi^{t,n}_{dyn}$ is $(\mathcal{U},\overline{\mathcal{U}})$-reducible. Therefore, for every choice of n, $\Phi^{t,n}_{dyn}$ is $(\mathcal{U},\overline{\mathcal{U}})$-reducible by Lemma VIII.2. These results then yield us, for each n, a corresponding map on Q:

$$\begin{aligned}&\Phi^{t,n}_{kin}(q) \overset{def}{=} \tau_Q \circ \Phi^{t,n}_{dyn}(q) = \\ &\left(\Phi^{\delta_1\ {}^{1}X\frac{t}{n}} \circ \Phi^{\delta_2\ {}^{2}X\frac{t}{n}} \circ \cdots \circ \Phi^{\delta_m\ {}^{m}X\frac{t}{n}}\right)^n(q)\end{aligned} \tag{11}$$

where each $\Phi^{\delta_i\ {}^{i}X\frac{t}{n}}$ is the flow of equations that are $(\mathcal{U},\overline{\mathcal{U}})$-reductions (as in Eq. (21)) from equations that generate the flow $\Phi^{\delta^i_i\mathcal{Y}\frac{t}{n}}$. Moreover, from Theorem VIII.3 we know that $\lim_{n\to\infty}\Phi^{t,n}_{kin}$ exists and that its limit is a solution to

$$\dot{q} \in \mathbf{X} \tag{12}$$

where $\mathbf{X} = co\{{}^{l}X | l \in L\}$ and the $\{{}^{l}X\}$ come from the reduced equations in Eq. (21). Therefore, part *i)* of Definition VIII.2 is satisfied.

(ii) The analysis of this second condition uses the same essential steps as above, but begins with the solution to the kinematic equations and works towards a dynamic solution. Starting with the solutions from Eq. (21), we know that for an individual model with index l we have $\dot{q}^i = u^a\ {}^{l}X^i_a$, or in vector form:

$$\dot{q} = u^a\ {}^{l}X_a. \tag{13}$$

Therefore, this MMDA system can be associated with governing equations having the form of Eq. (12). Again, for any given solution Φ of Eq. (12) we have $\frac{d}{dt}\Phi \in \mathbf{X}$, so we can choose a selection $s(\mathbf{X})$ such that $\Phi^{s(X)}$ locally approximates the flow for that solution. As before, we construct a sequence of solutions converging to $\Phi^{s(\mathbf{X})}$. By construction, there exists a $\Phi^{t,n}_{kin}$ whose limit is $\Phi^{s(\mathbf{X})}$.

From Def VIII.2 we know we must show there exists an η solution with

$$\frac{d}{dt}\Phi^{s(\mathbf{X})} = \eta.$$

By our construction, we know that

$$\lim_{n\to\infty}\Phi^{t,n}_{kin} = \Phi^{s(\mathbf{X})}(q_0, t)\ .$$

From part **(i)** above, for every n and $\Phi^{t,n}_{kin}$ there exists a corresponding $\Phi^{t,n}_{dyn}$ such that $\Phi^{t,n}_{kin}(q) = \tau_Q \circ \Phi^{t,n}_{dyn}(q)$. In the limit,

$$\lim_{n\to\infty} \Phi^{t,n}_{dyn} = \Phi^{s(\mathbf{Y})},$$

for some selection of the differential inclusion $s(\mathbf{Y})$. Consequently, $\Phi^{s(\mathbf{Y})}$ is a solution to Eq. (8), again by Theorem VIII.3. Taking the derivative of both sides, we get (after repeated application of the chain rule)

$$\frac{d}{dt}\Phi^{s(\mathbf{X})} = \frac{d}{dt}\lim_{n\to\infty} \Phi^{t,n}_{kin} = \lim_{n\to\infty} \frac{d}{dt}\Phi^{t,n}_{kin}$$
$$= \lim_{n\to\infty} \Phi^{t,n}_{dyn} = \Phi^{s(\mathbf{Y})}$$

so part *ii)* is satisfied. This ends the proof. ■

Notice that the proof of Theorem VIII.4 relied heavily on specifically constructing a solution with the desired properties based on *known* solutions to the individual models comprising the multiple model system. This result shows that determining the kinematic properties of the individual models in a multiple model system is sufficient for determining the kinematic properties of the entire system.

IX. The PDM and $(\mathcal{U},\overline{\mathcal{U}})$ Reducibility

This section addresses the relationship between the models produced by the power dissipation methodology and the kinematically reducible states of a generic mechanical system. An informal restatement of this is the question: does the PDM produce equations of motion that are kinematic reductions of Euler-Lagrange equations? First, we derive a result that will be shortly used to show the relationship between PDM solutions and solutions of mechanical, second order, systems.

Proposition IX.1: Given a configuration manifold Q and a set of constraints $\omega^i(q)$ which span the cotangent space $T^*_q Q$, then the input distribution $D_{kin}(q)$ minimizing $\mathcal{D}(q)$ will always satisfy $D_{kin}(q) = Null(\Omega_{sat})(q)$ where $\Omega_{sat}(q)$ is the collection of $w_i(q)$ which satisfy $w^i(q)\dot{q} = 0$ for $\dot{q} \in D_{kin}$.

Proof: Suppose that this was not the case. Then there would exist $v \neq 0$ which minimizes D such that if ω^i_s are the constraints which are satisfied, then $v \in Null\{\omega^i_s\}$ and $v \notin D_{kin}$. This implies that for the choice of $u^k = 0\ \forall k$, v still minimizes D. However, because the $\{\omega^i\}$ span T^*Q, 0 is the unique minimizer since D is convex in $\dot{q}$. This contradicts the assumption that $v \neq 0$ and is a minimizer of $\mathcal{D}$. ■

This result roughly corresponds to the intuition that the minimum dissipation in any unactuated direction is to not move at all in that direction. We should comment that this can still lead to a solution of no motion in the group variables s–if the unactuated constraints dominate the motion, then the actuators will all slip.

Next we consider the case where we are given a metric G for some mechanical system and a set of constraints described by one-forms $\{\omega_j\}$. What are sufficient conditions for the resulting system to be $(\mathcal{U},\overline{\mathcal{U}})$ reducible? Lemma IX.2 gives one sufficient condition which is invariant with respect to the metric G, and is a simple corollary to the work found in [24].

Lemma IX.2: Given a "constraint distribution" $D_{con} \subseteq TQ$ which annihilates the constraints $\{\omega_j\}$ and an input distribution D_{dyn}, if $D_{dyn} = D_{con}$ the mechanical system described by $\nabla_{\dot{q}}\dot{q} = uY$ is $(\mathcal{U},\overline{\mathcal{U}})$ reducible.

Proof: Denote by ∇ the connection and by $\overline{\nabla}$ the constrained connection defined by the Lagrange-dÁlembert principle (see Lewis [23] for details of this construction). We know that

$$\overline{\nabla}_X Y \in D_{con}\ \forall\ Y \in D_{con}\ and\ X \in \mathcal{T}(\mathcal{M}),$$

which implies

$$\overline{\nabla}_X Y + \overline{\nabla}_Y X \in D_{con}\ \ \forall X, Y \in D_{con}.$$

This in turn implies by Theorem VIII.1 that $\nabla_{\dot{q}}\dot{q} = uY$ is $(\mathcal{U},\overline{\mathcal{U}})$ reducible. ■

Therefore, $(\mathcal{U},\overline{\mathcal{U}})$ reducibility of a multiple model mechanical system is guaranteed *regardless of the metric G* when the constraint distribution is covered by the input distribution. Moreover, we already know that the power dissipation model only admits solutions where this is true. This allows us to interpret the use of the power dissipation method. The power dissipation method is a way of choosing a more tractable *subset* of contact states from the full Lagrangian contact mechanics. In other words, when we make the "quasistatic" assumption, we are merely restricting our attention to $(\mathcal{U},\overline{\mathcal{U}})$ reducible systems. Moreover, when the reaction forces due to friction do not lie in D_{kin}, then those contact states are not $(\mathcal{U},\overline{\mathcal{U}})$ reducible. However, we should be very clear that this only shows that the power dissipation method captures $(\mathcal{U},\overline{\mathcal{U}})$ reducible states when $D_{con} = D_{kin}$. That is, the correspondence only goes one direction: all PDM contact states are kinematic states, but not all kinematic states can be predicted by the PDM. There are examples of mechanical systems which are $(\mathcal{U},\overline{\mathcal{U}})$ reducible by virtue of properties of the metric G. For examples of such systems, see Lewis [23].

In summary, we have shown is the following.

Theorem IX.3: Given a configuration manifold Q with tangent space TQ and constraints represented by one-forms ω^i, then all solutions to the PDM are $(\mathcal{U},\overline{\mathcal{U}})$ reductions of solutions to the Euler-Lagrange equations on TQ constrained by a subset of $\{\omega^i\}$.

We should also remark on the relationship between Theorem VIII.1 (reduction for smooth systems) and Theorem VIII.4 (reduction for multiple model systems). In the smooth case, $(\mathcal{U},\overline{\mathcal{U}})$ reducibility is equivalent to geodesic invariance (for details, see Lewis [23]). However, in the nonsmooth case there is no well defined notion of geodesic invariance because the metric changes over time. Nevertheless, we were able to extend the notion of $(\mathcal{U},\overline{\mathcal{U}})$ reducibility relatively easily. Therefore, the concept of $(\mathcal{U},\overline{\mathcal{U}})$ reducibility is in some sense more general than that of geodesic invariance.

X. Examples

To illustrate how the results presented in this paper are useful, and point towards more general applications of theories developed here, we now revisit the examples from Section II. First, we come back to the bicycle example to illustrate all of the theory details. We study the bicycle example in detail as illustration, and then quickly summarize several applications in other related work. For instance, we show how this analysis helps to establish controllability characteristics for the Mars rover family of vehicles and stability analysis for distributed manipulation problems. We end this section with a brief discussion of how the method presented here can be applied to grasping and locomotion.

A. Bicycle

Now, we return to the bicycle example of Section II in detail. Assume that the bicycle is constrained to move on a line. Using the mechanics formulation as described in the Appendix, the configuration space is $\{x,\phi_1,\phi_2\}\in\mathbb{R}\times S^2$, and the Riemannian metric describing the kinetic energy is

$$G=(m+2J)dx\otimes dx+Jd\phi_1\otimes d\phi_1+Jd\phi_2\otimes\phi_2.$$

The two non-rolling constraints are

$$\dot{x}-R\dot{\phi}_1=0 \qquad \dot{x}-R\dot{\phi}_2=0$$

and the constraint covectors can be written as

$$\omega_1=dx-Rd\phi_1 \qquad \omega_2=dx-Rd\phi_2$$

As inputs, we have

$$F^1=d\phi_1 \qquad F^2=d\phi_2.$$

Now, for each combination of slipping and no slipping of the wheels, we have a set of equations to solve for. Therefore, we have four sets of equations to solve. Moreover, because the Christoffel symbols Γ^i_{jk} are all identically zero for this problem, the equations depend entirely on the input forces and external forces due to friction.

1) No slipping: When both wheels do not slip, the system must satisfy $\dot{\phi}_1=\dot{\phi}_2$. This, in turn, implies that the constraint distribution is 1-dimensional, spanned by

$$R\frac{\partial}{\partial x}+\frac{\partial}{\partial\phi_1}+\frac{\partial}{\partial\phi_2}.$$

Moreover, one can readily compute that the orthogonal complement of D is

$$span\left\{-\frac{J}{mR}\frac{\partial}{\partial x}+\frac{\partial}{\partial\phi_2},-\frac{J}{mR}\frac{\partial}{\partial x}+\frac{\partial}{\partial\phi_1}\right\}.$$

The associated input vector fields are

$$Y_1=Y_2=\frac{1}{2J+mR^2}\left(R\frac{\partial}{\partial x}+\frac{\partial}{\partial\phi_1}+\frac{\partial}{\partial\phi_2}\right)$$

and the equations of motion are therefore:

$$\ddot{q}=Y_1u^1+Y_2u^2.$$

It is easy to see that $\langle Y_1,Y_2\rangle=0$, so this is a kinematic system (that is, it is reducible to Eq. 4).

2) One wheel slipping: In the case where one wheel slips, we may assume without loss of generality that the slipping wheel is wheel number 1. In this case, the constraint distribution is

$$span\left\{R\frac{\partial}{\partial x}+\frac{\partial}{\partial\phi_1},\frac{\partial}{\partial\phi_2}\right\}.$$

Moreover, one can readily compute that the orthogonal complement of D is

$$-\frac{J}{mR}\frac{\partial}{\partial x}+\frac{\partial}{\partial\phi_1}.$$

To compute the reaction force due to the other wheel slipping, note that such a reaction force can be considered an external force, and can therefore be added to the right hand side of Eq. (20) with the associated control assuming constant unity value $u^a\equiv 1$. The associated input vector fields and external force vector fields are

$$\begin{aligned}Y_1&=\tfrac{1}{2J+mR^2}\left(R\tfrac{\partial}{\partial x}+\tfrac{\partial}{\partial\phi_1}+\tfrac{\partial}{\partial\phi_2}\right)\\ Y_2&=\tfrac{1}{J}\tfrac{\partial}{\partial\phi_2}\\ E&=\tfrac{R^2F_2^R}{J+mR^2}\tfrac{\partial}{\partial x}+\tfrac{RF_2^R}{J+mR^2}\tfrac{\partial}{\partial\phi_1}-\tfrac{RF_2^R}{J}\tfrac{\partial}{\partial\phi_2}\end{aligned}$$

and the equations of motion are therefore:

$$\ddot{q}=Y_1u^1+Y_2u^2+E.$$

To determine whether this system is kinematically reducible or not, we first note that $\langle Y_1,Y_2\rangle$ is again identically zero. Moreover, note that although Theorem VIII.1 does not directly address the case of external forces, we can by direct inspection of Definition VIII.2 see that if

$E \notin span\{Y_i\}$ then the system cannot in general be reducible. However, if $E \in span\{Y_i\}$ and the $\{Y_i\}$ satisfy the conditions for reducibility, then the system is automatically reducible because the external forces are "covered" by the inputs. Therefore, we need only check that E lies in the span of Y_1 and Y_2. Indeed, $E \in span\{Y_1, Y_2\}$ for this example. Therefore, this system is kinematically reducible. Note that this property does not depend on the particular description of the reaction force, and is moreover invariant with respect to the reaction forces' differentiability.

3) Both wheels slipping: When both wheels slip, there are no constraints to enforce. In this case, the constraint distribution is identically zero and the orthogonal complement is trivially the entire tangent space. Moreover, we can compute the reaction force due to the wheels slipping to be $w_1(F_1^R)$ and $w_2(F_2^R)$. The associated input vector fields and external vector fields are

$$\begin{aligned} Y_1 &= \tfrac{1}{J}\tfrac{\partial}{\partial \phi_1} \\ Y_2 &= \tfrac{1}{J}\tfrac{\partial}{\partial \phi_2} \\ E &= \tfrac{F_1^R + F_1^R}{m}\tfrac{\partial}{\partial x} - \tfrac{RF_1^R}{J^2}\tfrac{\partial}{\partial \phi_1} - \tfrac{RF_2^R}{J^2}\tfrac{\partial}{\partial \phi_2} \end{aligned}$$

and the equations of motion are therefore:

$$\ddot{q} = Y_1 u^1 + Y_2 u^2 + E.$$

In this case, it is clear that $E \notin span\{Y_1, Y_2\}$. Therefore this system (not surprisingly) is not kinematically reducible, at least for generic F^R.

B. Simplified Mars Rover

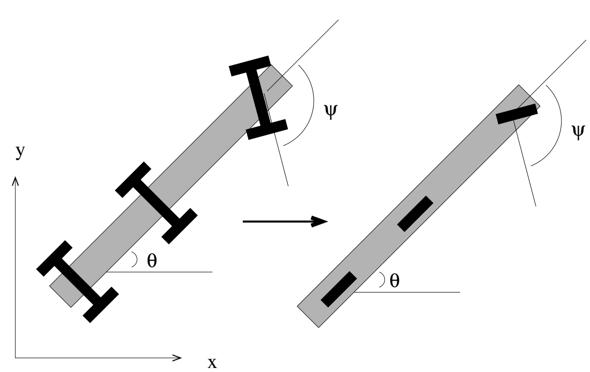

Fig. 2. Simplified Rocky 7. a) Is a cartoon of a six wheeled rover, and b) is a cartoon of a simplification of the rover.

Next we revisit the example of Fig. 1(b), whose geometry we simplify here for the sake of discussion. This simplification has three wheels, with all three wheels driven. This model can be interpreted as a simplification of the Mars rover Rocky 7 vehicle, also seen in Fig. 1. The three wheeled vehicle seen in the cartoon has a configuration space consisting of $(x, y, \theta, \psi, \phi_1, \phi_2, \phi_3)$. This system has six nonholonomic constraints (one associated with each wheel having both a no roll constraint and a no sideways slip constraint). Therefore, there are $2^6 = 64$ possible models governing the dynamics of the vehicle. For this reason, we do not relate all the calculations for this vehicle. However, it is easy to show, using a symbolic mathematics package such as *Mathematica*, that this system also has a subset of kinematic solutions, and that these solutions correspond to the the solutions to the PDM for this system. There only exist $\binom{6}{3}$=20 kinematic solutions for this system. Such a correspondence is important because the power dissipation method is very straight forward to solve and these solutions can be used for both controllability analysis and for purposes of motion planning (we have carried out this analysis in [32], [33]).

In [32], [33] we showed that this system's controllability properties can be analyzed using a set-valued extension of the Lie bracket (the prerequisite calculation for understanding controllability using the classical Lie Algebra Rank Condition (LARC)) that arises naturally in MMDA analysis. Controllability is important for systems like the Rocky 7 primarily because many motion planning algorithms for vehicles are based on controllability properties. For instance, Rapidly Exploring Random Trees (RRT) have been used with much success to develop motion planning strategies. However, the computational intensity of these calculations is formidable, and recently [10] showed that significant advantage can be taken by reducing mechanical systems to kinematic ones when using RRTs for motion planning. Work is currently underway to extend RRTs to the multiple model systems of this paper. See [28] for a preliminary motion planning that is based on the MMDA structure found here.

We should comment on the relationship between kinematic reducibility results and controllability results which can be obtained for multiple model systems [32], [33]. One of the intuitive aspects of Theorem VIII.4 is precisely that it is sufficient for each model to be $(\mathcal{U},\overline{\mathcal{U}})$ reducible in order to guarantee that the multiple model mechanical system is $(\mathcal{U},\overline{\mathcal{U}})$ reducible. That is, piecewise $(\mathcal{U},\overline{\mathcal{U}})$ reducibility is enough to guarantee $(\mathcal{U},\overline{\mathcal{U}})$ reducibility across discontinuities. However, in the case of controllability, this no longer holds. An MMDA system can switch among individually controllable systems in such a way as to destroy controllability [33]. Thus, controllability of each model in an MMDA is not sufficient for overall controllability.

The fact that there is such a high number of models for the Rocky 7 suggests the need for a reduction theory for multiple model systems. Indeed, for a six-wheeled system like the actual Rocky 7, there are $2^{12} = 4096$ possible models governing its dynamics, a completely unmanageable number. For the three wheeled vehicle in the cartoon, 20 kinematic models is also perhaps an unreasonably large number of models to analyze. In [33] we did an adhoc reduction of this model which turned it into a two model multiple model system (although it can be shown that no additional reduction is possible). Combining kinematic reduction with this multiple model reduction reduced the number of models from 4096 to 2. Therefore, formally utilizing reductions (both discrete and continuous) to reduce the dimensionality of the problem will be very useful, both for motion planning and estimation purposes. This will be a focus of future research.

C. Distributed Manipulation with Changing Contacts

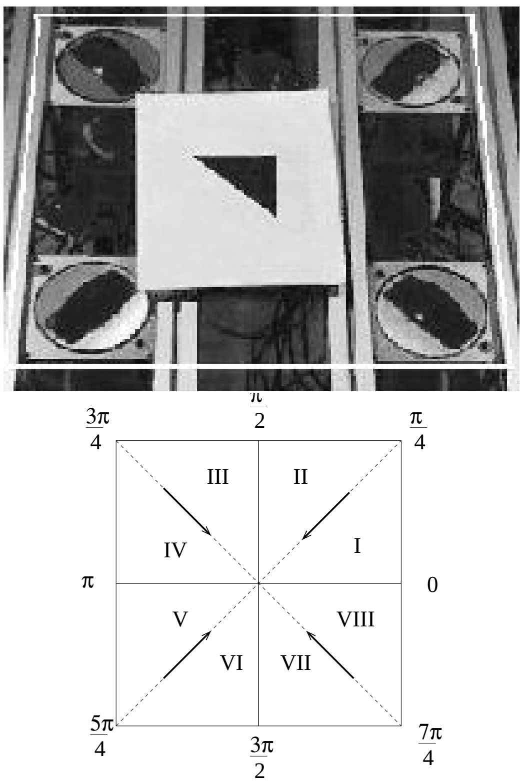

Fig. 3. Photograph and cartoon of 4 cell distributed manipulator.

Figure 3(a) shows a photograph of a particular configuration of a distributed manipulation experiment at Caltech pictured in Fig. 1(c) which has been used previously to test algorithms for distributed manipulation [34].[4] In the photograph we see four driving wheels whose rims are oriented towards the origin. Each actuator is a one degree of freedom actuator. We use a piece of plexiglass (for purposes of visualization) on top of the four wheels to represent a manipulated object. The white line seen in the photograph indicates the outline of the plexiglass. The goal is to control the center of mass to the origin in $\mathbb{R}^2$ with a desired orientation of $\theta = 0$. To do this, we obtain feedback of the plexiglass' configuration by affixing a piece of paper with a black triangle (also seen in the photo) whose right angle corner coincides with the plexiglass' center of mass. Using this, we obtain the position and orientation of the plexiglass through visual feedback. Figure 3(b) is a cartoon of the experiment, where the four arrows correspond to actuators and the regions denoted by **I-VIII** and **0-$\frac{7\pi}{4}$** will be important in our subsequent description of the equations of motion described by the PDM.

Note that this system thus described is overactuated because there are four inputs and only three outputs. Assume the coefficient of friction is the same for all four driving actuators. In this case we can show that the model switches as the center of mass moves across the array. In fact, under these assumptions, the actuator wheel nearest to the center of mass will have both its "rolling" constraint and its "sideways" slip constraint satisfied. The actuator wheel second closest to the center of mass will have one of its two constraints satisfied. In the case of the wheels shown in the figure, it will be the rolling constraint. For details on this analysis, see [31]. Denote the actuator input associated with the closest actuator by u_i and the actuator input associated with the second closest actuator by u_j. Then these considerations lead to first order governing equations of motion of the form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = g_1 u_i + g_2 u_j \tag{14}$$

where

$$g_1 \in \begin{bmatrix} \frac{-y_i}{(x_j-x_i)\sin(\theta_j)+(y_i-y_j)\cos(\theta_j)} \\ \frac{x_i}{(x_j-x_i)\sin(\theta_j)+(y_i-y_j)\cos(\theta_j)} \\ \frac{u_j}{(x_i-x_j)\sin(\theta_j)+(y_j-y_i)\cos(\theta_j)} \end{bmatrix} \tag{15}$$

$$g_2 \in \begin{bmatrix} \frac{\sin(\theta_j)((x_i-x_j)\cos(\theta_i)+y_i\sin(\theta_i))+\cos(\theta_i)\cos(\theta_j)y_j}{(x_j-x_i)\sin(\theta_j)+(y_i-y_j)\cos(\theta_j)} \\ \frac{-\cos(\theta_i)\cos(\theta_j)x_i-\sin(\theta_i)(x_j\sin(\theta_j)-(y_i-y_j)\cos(\theta_j))}{(x_j-x_i)\sin(\theta_j)+(y_i-y_j)\cos(\theta_j)} \\ \frac{-\cos(\theta_i-\theta_j)}{(x_i-x_j)\sin(\theta_j)+(y_j-y_i)\cos(\theta_j)} \end{bmatrix} \tag{16}$$

In these equations x_i, y_i, and θ_i refer to the planar coordinates and orientation of the i^{th} actuator. The set-valued notation of (15) and (16) refers to the fact that at a transition between actuators i and j being the two closest actuators to actuators k and l being the closest the kinematics are discontinuous. Therefore, at these points we must allow multi-valued differentials in order to guarantee existence of solutions to the differential equation in (14). See [27] for more details. It should be noted that here the index notation should be thought of as mapping (i, j) pairs to equations of motion in some neighborhood (not necessarily small) around the i^{th} and j^{th} actuator. In each

[4] Video of these experiments can be found at the website *http://robotics.caltech.edu/∼murphey*.

region $I - VIII$ the kinematics are smooth, but when a trajectory crosses a boundary **0-$\frac{7\pi}{4}$**, there is a discontinuity in the kinematics. It is possible to obtain point stabilization to $(x, y, \theta) = (0, 0, 0)$ from any initial condition using discontinuous control laws based on the kinematics and knowing the current model (see [27] for details of this control design). Moreover, this stability is provably exponential. However, there are many questions relevant to this system which remain unanswered. In particular, we are currently developing algorithms which do not require any knowledge of the slipping state, and instead use an online estimation process based on hierarchical control like that found in [4], [19], [20], [18].

D. Relationship to Grasping and Locomotion

We briefly give our vision of how the preceding ideas can be related to both grasping and locomotion. Traditionally, analysis of grasping and locomotion has assumed clean interactions between the robot and its environment. Moreover, kinematic analysis has proven to be a very computationally and theoretically useful venue for understanding many issues in both areas. However, in real robotic systems, interactions in contact are often not clean, and we expect slipping to take place. Consider, for example, the hand shown in Fig. 1. As the hand manipulates the ball, its fingers will slip against the surface. However, we generally expect such motions to not interfere with the stability of the motion. The analysis presented in this paper provides a forum for robustness analysis as well as development of algorithms that explicitly require slipping.

XI. Some Final Remarks

In this paper we derived conditions that are both necessary and sufficient for a multiple model system to be kinematically reducible. Such an understanding of a system's kinematic motions is important for the purposes of tasking and motion planning. The structure we describe here is put to advantage in [34] in an application to distributed manipulation and in [33] where we analyze the controllability properties of an example like that found in Fig. 1. Moreover, it has future potential for greatly simplifying friction compensation problems in robotics. The notion of kinematic reducibility we presented can be related to the Power Dissipation Method, a method for determining the quasistatic equations of motion for an overconstrained system (see [3], [39]). We have been able to show that the solutions to the power dissipation method correspond to kinematic solutions of multiple model systems.

We do not claim that the PDM is a better model than the full Lagrangian setup, only that it is more tractable. It produces first order equations of motion that are amenable to analysis. Moreover, the fact that it allows us to compute explicit controllers that work on a real experiment is an indication of its validity [34]. Nevertheless, there are certainly important systems that must be treated in the full Lagrangian mechanical framework, since even in the example of the planar bike there are important dynamic states not accounted for in the PDM. This determination will in general have to be made by the control designer.

Lastly, this work leaves several open questions to be answered. First of all, in the definition presented in this paper the dissipation functional is only applicable to a finite number of contacts. However, in many pushing problems the frictional interaction occurs at the interface between two continuous media. The example of the Mars rover in Section X-B makes it clear that reduction theory (beyond kinematic reduction theory presented here) needs to be formally explored for multiple model systems. Lastly, there is the question of external forces. Our use of kinematic reducibility in the example avoids the problems of differentiation of friction forces because the manifold structure provides all the information we need. However, this cannot be expected in general, and there is a clear need to extend the work in [23] to cases that generic reaction forces entering the equations of motion.

Appendix

We assume the reader is familiar with the basic notation and formalism of differential geometry and nonlinear controllability theory. See [35], [40], [1], [7], [43] for more details.

The notion of $(\mathcal{U},\overline{\mathcal{U}})$-reducibility formalizes what is meant by kinematic reducibility. For mechanical systems, we consider inputs $u : [0, T] \to \mathbb{R}^m$ that are essentially bounded and Lebesgue integrable. In Lewis [23], it was assumed that inputs are absolutely continuous functions, since piecewise continuity implies that instantaneous changes in system velocity are possible. In the presence of inertial effects, such changes can only occur when infinite forces are allowed. We keep this assumption on the inputs. However, here state transitions are being *approximated* with piecewise continuous signals. This is a common approximation in many areas of physical modeling [42]–, such as impacting bodies. Therefore, we only require that absolute continuity hold locally rather than globally.

Definition .1: $f : [a, b] \to \mathbb{R}^m$ is *absolutely continuous* if for each $\epsilon > 0 \; \exists \; \delta > 0$ such that for every finite collection $\{(t_i, t_i^{'})\}_{1 \leq i \leq N}$ of non-overlapping intervals in $[a, b]$ with the property that

$$\sum_{i=1}^{N} |t_i^{'} - t_i| < \delta \quad we \quad have \quad \sum_{i=1}^{N} \|f(t_i^{'}) - f(t_i)\| < \epsilon$$

This definition implies that Df exists almost everywhere.

Like Lewis [23], we restrict our attention to systems that can be modeled as *simple mechanical systems* in a piecewise sense. In simple mechanical systems, the Lagrangian takes the form $\mathcal{L} = K.E. - V$. Assume that Q is an n-dimensional configuration manifold, and G is a Riemannian metric on Q defining the kinetic energy. Since many of the applications of interest are systems with no potential energy, let us simplify to the case where $\mathcal{L} = K.E.$ (i.e., $V = 0$). Denote by v_q elements in the tangent space of Q at q, T_qQ. With zero potential energy, the system Lagrangian takes the form $\mathcal{L} = \frac{1}{2}g(v_q, v_q)$.

Definition .2: The *Christoffel symbols* for the Levi-Civita connection ${}^g\nabla$ (associated with the metric G) are

$$\Gamma^i_{jk} = \frac{1}{2}G^{il}\left(\frac{\partial G_{jl}}{\partial q^k} + \frac{\partial G_{kl}}{\partial q^j} - \frac{\partial G_{jk}}{\partial q^l}\right) \tag{17}$$

where summation over repeated indices is implied used unless otherwise stated, and upper indices indicate the inverse.

Definition .3: In coordinates, the *covariant derivative* of Y with respect to X is

$${}^G\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j}X^j + \Gamma^i_{jk}X^jY^k\right)\frac{\partial}{\partial q^j} \tag{18}$$

Definition .4: The *symmetric product* between two vector fields X and Y is defined to be

$$\langle X : Y\rangle =^G \nabla_X Y +^G \nabla_Y X \tag{19}$$

With these definitions in mind, we can quickly summarize appropriate notions of dynamic and kinematic mechanical systems. Given a metric G on the manifold Q and inputs u^a, it is possible to show that the Euler-Lagrange dynamical equations can be written in the form:

$${}^G\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t)) \tag{20}$$

where $t \to c(t)$ is a path on Q and $c'(t) = \frac{d}{dt}c(t)$. On the other hand, given input velocities $\overline{u}^\alpha$, *kinematic* equations can be written in the form:

$$\dot{q}(t) = \overline{u}^\alpha(t)X_\alpha(q(t)) \tag{21}$$

Theorem VIII.1 relates Eq (20) to Eq (21). As noted in Lewis [23], the symmetric product plays a similar role in establishing $(\mathcal{U},\overline{\mathcal{U}})$ reducibility to the Lie bracket in establishing integrability. Some other things to note about kinematic reducibility include the following. First, all fully actuated systems are automatically kinematically reducible because their dynamic input vector fields are always closed under symmetric products. For instance, the forward kinematics of a robotic manipulator are kinematic whether moving in air (where the kinematic approximation is obvious) or in a viscous fluid of some sort. Also, kinematic reducibility is not the same thing as the "quasistatic" assumption commonly made in robotics. This is because kinematic reducibility only requires that there be a complete correspondence between dynamic motions and kinematic motions, whereas "quasistatic" assumptions, when formalized at all, typically require that the system be moving slowly in some sense. As noted in [39], the quasistatic case can only be equated to Newton's laws when the friction is Coulombic, but we note that here kinematic motions are independent of friction model. This fact seems to have reasonably deep implications for friction compensation, and will be the topic of future study.

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