Variational Methods for Contact Mechanics

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***Abstract*— This paper provides an overview of recent progress on discrete variational integration for constrained mechanical systems experiencing elastic, plastic, and simultaneous impacts. We discuss several methods and their characteristics in terms of convergence and what the conserved quantities are.**

I. INTRODUCTION

This brief paper introduces some recent results regarding analyzing the nonsmooth behavior of impacting systems using discrete variational mechanics. We provide an overview of recent work that utilizes the mechanical variational principle at the time of impact to generate numerical predictions for an impacting system. In contrast to complementarity formulations of contact, where Newton's equations are generalized to include impulses and then the complementarity conditions help one solve for discrete time (possibly impulsive) forces that satisfy the inequality constraints, variational analysis generalizes stationarity of the action principle to the nonsmooth case. (Solutions, though obtained from the variational formulation, will satisfy the complementarity conditions.) These variational algorithms are well-posed for constrained, forced systems and can be used for elastic as well as plastic impacts. We end with a discussion of computational scalability and software implementation.

II. VARIATIONAL INTEGRATION FOR SMOOTH SYSTEMS

The use of variational methods for mechanics goes back to Lagrange, and involves the introduction of the Lagrangian L, typically the kinetic energy KE minus the potential energy V (all expressed in terms of generalized coordinates q). Along with constraints $\phi(q) = 0$ and external forces F, the application of variational analysis to the action integral (the integral of the Lagrangian over the time interval) yields the Euler-Lagrange equations

$$\frac{d}{dt}D_2L(q,\dot{q}) - D_1L(q,\dot{q}) = F + \lambda D\phi(q)$$
$$\phi(q) = 0$$

where the second equation is typically differentiated twice and the system of equations is solved to get rid of the dependence on the Lagrange multiplier λ. This ordinary differential equation is then discretized and solved using standard methods, such as Runge-Kutta schemes of various orders (possibly with adaptive time-stepping).

Discrete mechanics approaches, on the other hand, discretize the Lagrangian directly to give the *discrete Lagrangian* L_d. Applying variational analysis to a discrete action sum (that approximates the action integral) yields the discrete Euler-Lagrange equations

$$D_2L_d(q_{i-1}, q_i) + D_1L_d(q_i, q_{i+1}) = \lambda D\phi(q) + F_d$$
$$\phi(q_{i+1}) = 0.$$

Note that this is now a discrete time rootsolving problem for q_{i+1}, given q_i and q_{i-1}. This discrete-time representation has the advantage of preserving the discrete momentum, the symplectic form, and has guarantees on energy behavior. Moreover, there exists a *modified* system (where the Hamiltonian is no longer the original Hamiltonian, but a perturbation of it) that this discrete-time representation exactly solves.

The goal of the work described below is to bring the same structure-preserving characteristics present in smooth discrete mechanics to the analysis and computation of nonsmooth impact equations. With each approach we start with elastic impacts and then treat plastic impacts. The next few sections describe several approaches as well as their positive and negative attributes.

III. THREE ALGORITHMS FOR NONSMOOTH CONTACT

This section describes three different ways of computing an impact. We classify the methods by the discrete quantification of energy that is conserved when simulating elastic collisions. The three approaches are a) conserving the continuous time energy, b) conserving a discrete-time energy, and c) conserving the energy of the modified Hamiltonian. These approaches are each reasonable, but they have different characteristics in terms of what they conserve and their computational costs, as listed in Table I.

Continuous Time Energy Conservation (CTEC)

The CTEC method (the basic idea of which can be found in many works including [2], [6]) is defined by enforcing conservation of the continuous time definition of energy at the discrete time node that marks the impact. This idea is linked to hybrid systems simulation methods in that the simulation task is divided into sequences of continuous simulation and discrete instances of event resolution (marked by guard detection, and state reset). The CTEC method has the lowest computational and implementation cost, given that evaluating the CTEC impact map is explicit for any system with a quadratic kinetic energy[1]. In fact, the CTEC impact

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[1] finding an impact time is an implicit calculation, but this is true for all three methods

Method	CTEC	DTEC	MHC
Symplectic	No	Yes	Sometimes
Conservation of Momentum	Yes	Yes	Yes
Scalar Conserved Quantity	CE	DE	MH
Implementation	Easy	Hard	Harder
Impact map	Explicit	Implicit	Implicit
Computational cost	Low	Medium	Medium-High
Dense impact behavior	Good	Bad	Good
L^2 Error	Highest	Med	Lowest
Structured L^2 Convergence	Yes	No	Yes

TABLE I

TABLE OF CHARACTERISTICS OF THREE APPROACHES TO RESOLVING IMPACTS IN DISCRETE TIME. (CE=CONTINUOUS ENERGY, DE=DISCRETE ENERGY, MH=MODIFIED HAMILTONIAN)

map is so cheap to implement it is often used to provide initial guesses to the implicit routines of the other impact simulation methods. Also, the CTEC impact map has the most straightforward extension to the multiple impact case.

In terms of disadvantages, the CTEC method conserves a local definition of energy and thus is susceptible to drift in energy in the overall simulation due to the impacts (this can be seen in systems as simple as the double pendulum, the energy behavior of which can be seen in Fig.1). CTEC has no overall notion of symplecticity for nonsmooth trajectories (although the smooth integration flow and impact map each have an associated symplectic form—but they are not the same). Lastly, in one specific simulation of the double pendulum the CTEC method performed the worst of all three methods in terms of the L^2 norm of the error in the discrete trajectory relative to a benchmark simulation [4]. (This third place finish was uniform across a range of timesteps, but may have been somehow related to the initial condition used in the simulation.)

Discrete Time Energy Conservation (DTEC)

The DTEC method [3] conserves a variational discrete time definition of energy through impacts. As the method stems from a discrete time Hamilton's principle, it conserves one uniform symplectic form along the entire nonsmooth trajectory. For all but the simplest cases this method is implicit and thus requires the second most computational effort (more than CTEC, but less than MHC, discussed next). In terms of disadvantages, DTEC (similar to CTEC) conserves a local definition of energy and thus is susceptible to overall drift in energy due to impacts (again seen in the double pendulum energy in Fig.1). Furthermore, it has been analytically demonstrated that accuracy of the DTEC impact law can vary wildly (e.g. the method can fail completely) with increasing time steps or density of impacts. In the simulation of the double pendulum the CTEC method performed the second best of the three methods in terms of the L^2 norm of the error in the discrete trajectory relative to a benchmark simulation [4]. However, DTEC was the only one of the three methods that did not display structured quadratic convergence in the L^2 norm under varying timesteps. This lack of structure makes the method ill-suited for automated time step selection schemes.

Modified Hamiltonian Conservation (MHC)

The MHC method [4] conserves a modified Hamiltonian (MH) through impacts. The definition of this MH stems from the existing backwards error analysis associated with variational integration of smooth dynamics. By formulating the MHC impact law to respect the backwards error analysis, it is guaranteed that the output of an MHC simulation approximates, and sometimes exactly provides, a discrete sampling of the exact solution of a 'nearby' nonsmooth Hamiltonian system, leading to excellent energy behavior (seen in Fig.1). Though it is not necessarily required, MHC can be modified to eliminate the use of local information in its impact map. As such, this method does not experience the energy drift seen (on occasion) with the other two methods. In the simulation of the double pendulum the MHC method provided the lowest L^2 norm of the error in the discrete trajectory relative to a benchmark simulation and quadratic convergence under varying timesteps [4]. In terms of disadvantages, MHC requires the most computational and implementation effort.

Discussion of three methods

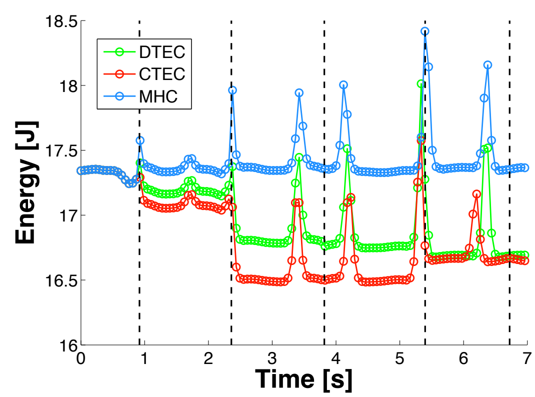

Fig. 1. Comparison of CTEC, DTEC, and MH techniques using a double pendulum impacting a vertical surface as a benchmark.

Figure 1 shows the three methods as applied to a double pendulum striking a wall. The MH methods shows the best behavior while the CTEC method performs relatively badly. The key thing is that these three methods, each equally reasonable, lead to three very different algorithms, with different complexity, accuracy, and convergence properties. It seems clear that is one wants the "best" representation of the dynamics, MHC performs the best, as seen in Fig.1. If, however, one wants an adequate representation of the impact dynamics at low computational cost, CTEC appears the best choice. For our work, we assume that physical accuracy is the most important characteristic, so the added cost of MHC is acceptable.

We next discuss the basic approach we have taken in applying discrete variational analysis to the cases of simultaneous impact and plastic impact.

Plastic Impacts

In order to treat plastic impacts, variations are evaluated such that they are constrained to be tangent to the impact surface at the time of impact. This leads to a physical model that is not meaningful for the continuous time system where the constraint forces must be impulses, but for the discrete variational system those forces approximate the Zeno behavior over a small time step. An important thing to note is that detecting a plastic impact is only possible when using the CTEC and the MHC methods, since the DTEC method becomes unreliable when more than one impact occurs per time step and therefore cannot predict Zeno behavior. Lastly, the variational methods we discuss here are guaranteed to solve the associated complementarity problem, but they narrow down the set of solutions more than the complementarity problem does.

Simultaneous Impacts

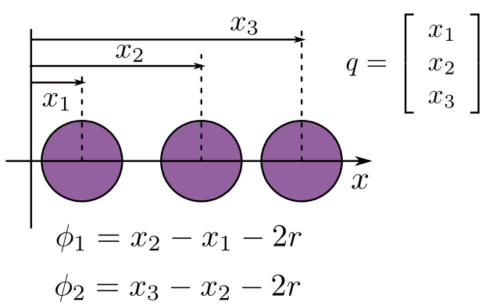

Fig. 2. Simultaneous impacts have nonunique outcomes in general, but in the case of some systems such as Newton's cradle (where impacts occur when $\phi_1 = \phi_2 = 0$) variational analysis predicts unique outcomes.

Simultaneous impacts occur when two or more contacts are present at the same time. Evaluating such impacts with an elastic method is difficult since conservation laws do not (and cannot) give a unique answer is most cases. The only certain way to consistently obtain a unique solution in such cases is to assume a plastic impact model ahead of time. This creates difficulties in modeling systems like Newton's cradle, where perfectly elastic impacts have characteristics of plastic impacts (e.g., once contact is made, it is maintained for a finite interval). Instead, we use a sequential impact method which guarantees a finite number of solutions and, in some cases, unique solutions [5]. While we have only implemented this method using CTEC it could easily be extended to the MHC impact method.

IV. Scalability

What if one wants to compute the impact laws discussed above for a complex system such as the hand model shown in Fig.3? We have implemented the discrete variational equations in the introduction for smooth, forced, constrained systems in open source software `trep`[2] The algebraic quantities required for computing simulations and optimal control laws—primarily multiple orders of derivatives of the discrete Lagrangian L_d with respect to configurations—are the same for smooth systems as they are for impacting systems. Moreover, `trep` calculates these derivatives *exactly*, for arbitrary interconnected mechanical systems. Hence, our next task is to implement one of the impact strategies mentioned above and apply it to example simulation and optimal control problems such as the hand as it makes and breaks contact.

[2]Available at *http://trep.sourceforge.net*.

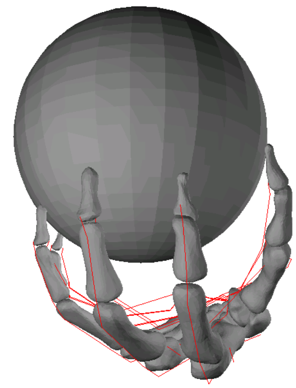

Fig. 3. It is not self-evident that the techniques discussed thus far apply to a more complex system like the many degree-of-freedom hand shown above. The `trep` software package provides the needed data to resolve the variational equations to second-order.

V. Conclusions and Future Work

The choice of impact representation largely depends on how much computation one is willing to use to get a good representation of the impact. Using the modified Hamiltonian, the MH method seems to be the best method overall, but the calculation is also the most expensive. Therefore our `trep` software will likely use the MH method for representing elastic, plastic, and simultaneous impact. Moreover, within `trep` one can already compute nonlinear optimal control policies for the constrained hand pictured in Fig.3 and other similarly complex systems. We will use the tools we discuss here to implement algorithms that enable one to both simulate and control a complex system with impact dynamics.

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