Projection-Based Switched System Optimization

T. M. Caldwell and T. D. Murphey

Abstract— **The Pontryagin Maximum Principle is applied to the switched system optimization problem resulting in a generalization of a well known necessary condition for switching time optimality. The switched system optimization may be formulated as an infinite dimensional problem where the switching control design variables, at any given time, are constrained to the integers. This paper analyzes projection-based techniques for handling the integer constraint. The necessary condition derived in this paper uses the cost composed with the projection of the design variables onto the feasible set. A specific form of projection is considered and two candidate projections are proposed—one projects the immediate value of the switching control and neglects the state while the second is variable on the projected state error.**

I. Introduction

This paper is concerned with the problem of switched system optimization, for which the design variables include the mode sequence. The mode sequence may be represented by a switching control signal that is constrained to the integers. A common theme in switched system optimization is to relax, or embed, the set of switching controls and find the optimal of the relaxed cost, which results in chattering solutions [1], [12], [16]. The resulting approach is an infinite dimensional optimal controls problem with inequality constraints. A method for solving the problem is to discretize time and use SQP to optimize over basis functions that approximate the state and control signals [16]. Furthermore, [12] discusses digital implementation of the chattering solutions and considers a minimum dwell time restriction on the transition times. These methods, however, pursue infeasible solutions and attempt to back out feasible suboptimal ones. They also do not make use of switching time optimization, an efficient switched system optimization tool for when the modes are fixed [4].

Another common theme is to alternate between optimizing the switching times and updating the mode sequence [17]. Switching time optimization is well understood and efficient. Adjoint calculations for both the gradient [4] as well as the Hessian [3], [10] exist. Furthermore, when the modes are linear time-varying, a single set of differential equations independent of switching times and mode sequence may be solved such that switching time optimization for each mode sequence does not require additional integration [2]. The modal update is not as well understood, but mode injection schemes have been shown to converge [4], [6].

We use a projection operator so that the design variables are in an unconstrained space but the cost is still on the set of switched system feasible trajectories. The projection maps signals from the unconstrained space to the set of feasible switched system trajectories, wherein each feasible switched system trajectory corresponds to a switching schedule consisting of a mode sequence and set of switching times. This paper serves as an introduction to projection-based methods for switched system optimization. In future works, we will propose projection-based algorithms based on the two stage switching time optimization and mode sequence update methodology of [17].

The switched system optimization problem is constrained to the switched system dynamics, which includes a constraint to the integers. We, instead, consider an unconstrained problem where the cost function is variable on the projection of the design variables to the constrained set. Suppose $J(\cdot)$ is the cost, (x, u) are feasible state and control variables, (α, μ) are state and control variables in the unconstrained space and $\mathcal{P}$ is a mapping from the unconstrained space, $\mathcal{R}$, to the set of feasible switched system trajectories, $\mathcal{S}$. Then, the standard constrained optimization problem and the unconstrained problem are (respectively)

$$\arg\min_{(x,u)\in\mathcal{S}} J(x, u) \text{ and } \arg\min_{(\alpha,\mu)\in\mathcal{R}} J(\mathcal{P}(\alpha, \mu)).$$

It is important to note that what we refer to as the unconstrained problem differs from, for example, finding the solution to a problem with unconstrained cost—e.g. $\min_{(\alpha,\mu)\in\mathcal{R}} J(\alpha, \mu)$—and projecting the solution to the feasible set. We are concerned with the problem that while the design variables, (α, μ), are unconstrained, the cost is still on the feasible set since $\mathcal{P}$ returns feasible state and control trajectories. In fact, when $\mathcal{P}$ is a projection, the solution to the constrained problem is equal to the projection of the solution to the unconstrained problem—i.e. $(x^\star, u^\star) = \mathcal{P}(\alpha^\star, \mu^\star)$. The same rationale is used in [8], in which a trajectory tracking projection operator is applied to the problem of nonlinear optimal controls.

The unconstrained space under consideration is the space of Lebesgue integrable functions from the interval $[0, T]$ to $\mathbb{R}^{n+N}$. As this space is open, if the optimal point, $\xi^\star := (x^\star, u^\star)$, exists, then it must be the case that the partial derivative of the Hamiltonian with respect to ξ is zero—i.e. $\frac{\partial}{\partial\xi} H(\xi^\star) = 0$. Due to the integer constraint, however, an optimal point in the constrained set may be on

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the boundary of the set, in which case, the partial derivative of the Hamilton with the control need not be zero [13].

We use a simple projection operator along with the condition $\frac{\partial}{\partial \xi} H(\xi^\star) = 0$ to obtain a generalization of the well-known necessary condition for optimality of switched systems. Refer to [15], [14] for the necessary condition for hybrid system optimization, in which the necessary condition for switched systems enters as a transversality condition. We provide the projection-based derivation of the switched system necessary condition in order to demonstrate the usefulness of the projection operator.

This paper is structured as follows: Section II presents switched system representations which will be used throughout the paper. Section III proposes candidate projection operators from the unconstrained space to the set of feasible switched system trajectories. The derivative of a specific form of projection is also given. Section IV applies the Pontryagin Maximum Principle along with what we refer to as the $\max$ projection to give a necessary condition on switched system optimization.

II. The Switched System

A switched system's evolution is given by its transitions between modes of operation. The control is the timing of the mode transitions and into which modes the system transitions. This section describes mathematical representations of the relationship between the state and the control.

Suppose $\mathcal{X}$ and $\mathcal{U}$ are spaces of Lebesgue integrable functions from the time interval $[0, T]$ to, respectively, $\mathbb{R}^n$ and $\mathbb{R}^N$. We label the space $\mathcal{R} = \mathcal{X} \times \mathcal{U}$ as the unconstrained space.

Consider a switched system composed of the N distinct equations of motion $f_i(x)$, $i = 1, \ldots, N$, which are $\mathcal{C}^r$, $r > 0$ on $\mathcal{X}$. Now, define the function

$$F(x, u) = \sum_{i=1}^{N} f_i(x) u_i. \tag{1}$$

Clearly, $F(x, u)$ is $\mathcal{C}^r$ on $\mathcal{X}$ and $\mathcal{C}^\infty$ on $\mathcal{U}$.

In order for the pair $(x, u) \in \mathcal{R}$ to constitute a valid switched system, both of the following must be true. First, x and u must satisfy $\dot{x} = F(x, u)$, i.e.

$$G(x, u, t) := x(t) - x(0) - \int_0^t F(x(\tau), u(\tau)) d\tau$$

equals zero for all $t \in [0, T]$. The integral is understood to be the Lebesgue integral. Second, u must be restricted as follows. Define $E^N = \{e_1, e_2, \ldots, e_N\}$, where e_i has value 1 at its i^{th} entry and 0 for every other entry. Define the set of switching control inputs as

$$\Omega = \{u \in \mathcal{U}^N | \forall t \in [0, T], u(t) \in E^N\}$$

Now, the set of switched systems $\mathcal{S} \subset \mathcal{R}$ is given as the set of all (x, u) such that

$$\begin{array}{ll} (i) & x \in \mathcal{X}, \\ (ii) & u \in \Omega \subset \mathcal{U}, \\ (iii) & G(x, u, t) = 0 \text{ for all } t \in [0, T]. \end{array} \tag{2}$$

Often, it will be useful to represent an element of $\mathcal{S}$ using its *switching signal* or *switching schedule* representations. For $u \in \Omega$, the equivalent switching signal is the $\sigma(t)$ such that $u(t) = e_{\sigma(t)}$. Clearly, the value of $\sigma(t)$, $t \in [0, T]$ is an integer from the set $\{1, 2, \ldots, N\}$. The switching schedule is the mode sequence and set of switching times, $(\Sigma, \mathcal{T}, M)$, defined as follows:

- $M - 1$ is the number of discontinuities of u,
- $\mathcal{T} = \{T_1, T_2, \ldots, T_{M-1}\}$ are the times, $0 < T_1 \le T_2 \le \cdots \le T_{M-1} < T$, for which u is discontinuous, and
- $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_M\}$ is the sequence of modes such that $u(t) = e_{\sigma_i}$ for $t \in (T_{i-1}, T_i)$.

III. The Projection Operator

We propose candidate projections from the unconstrained space, $\mathcal{R}$, to the set of feasible switched system trajectories, $\mathcal{S}$. We begin by investigating a general form of the projection by noting the binary constraint in Eq.(2) on the switching control limits the set of mappings the projection can take.

Consider the mapping $Q : \mathcal{S} \to \Omega$ where Q satisfies a *reproducing* condition—i.e. $\forall (x, u) \in \mathcal{S}$, $u = Q(x, u)$. The condition is so that the following can be a projection. Define the map $\mathcal{P} : \mathcal{R} \to \mathcal{S}$

$$(x, u) = \mathcal{P}(\alpha, \mu) := \left\{ \begin{array}{l} \dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0 \\ u(t) = Q(\alpha, \mu; t). \end{array} \right. \tag{3}$$

Here, and throughout the paper, we use the notation $Q(\alpha, \mu)$ for a curve in Ω while $Q(\alpha, \mu; t) := (Q(\alpha, \mu))(t)$ returns the value of the curve at time t. In order for $\mathcal{P}(\alpha, \mu)$ to be a projection—i.e. $\mathcal{P} \circ \mathcal{P}(\alpha, \mu) = \mathcal{P}(\alpha, \mu)$—$Q(\alpha, \mu)$ must be reproducing and $\dot{x} = F(x, Q(\alpha, \mu))$ must have a solution for each $(\alpha, \mu) \in \mathcal{R}$. Examples of when $\dot{x} = F(x, Q(\alpha, \mu))$ does not have finite escape time are the following:

- When each mode $f_i(\alpha)$, $i = 1, \ldots, N$, is linear.
- When $Q(\alpha, \mu)$ exponentially stabilizes the system. It is shown in [9] that $\dot{x} = F(x, Q(\alpha, \mu))$ is exponentially stable if each mode, $f_i(\alpha)$ $i = 1, \ldots, N$, is Lyapunov stable and the switching control, $Q(\alpha, \mu)$ satisfies an average dwell time condition. Furthermore, the modes could satisfy certain Lie algebraic structure [11].

In this paper, we will on occasion assume that the solution to $\mathcal{P}$ exists over the full time interval by referring to the following assumption.

Assumption 1: The mapping $\mathcal{P}$ defined by Q and the differential equation in Eq.(3) exists for each $(\alpha, \mu) \in \mathcal{R}$.

Suppose $\mathcal{P}$ exists and $u = Q(\alpha, \mu)$ for some $(\alpha, \mu) \in \mathcal{R}$. For any index $i = 1, \ldots, N$, the signal $u_i(\cdot)$ is binary, transitioning between the values 0 and 1. Therefore, we represent $Q_i(\alpha, \mu)$ as the composition of the step function with the function $a_i : \mathcal{R} \times [0, T] \to \mathbb{R}$ which dictates when the transitions occur. Consider $Q = [Q_1, \ldots, Q_N]^T$ of the following form

$$Q_i(\alpha, \mu; t) = 1(a_i(\alpha, \mu; t)) \tag{4}$$

where $1(\cdot)$ is the step function and the right side of Eq.(4) is understood as

$$1(a_i(\alpha,\mu;t)) = \begin{cases} 1 & a_i(\alpha,\mu;t) \geq 0 \\ 0 & \text{else.} \end{cases}$$

The functions a_i, $i = 1, \ldots, N$ must have specific properties so that Q is reproducing.

A. Properties of a_i

In order for $Q(\alpha, \mu)$ to be both reproducing and return an element of Ω, the signal $a = [a_1, \ldots, a_N]^T$ must satisfy the following:

1) At almost all $t \in [0, T]$, there is an $i = 1, \ldots, N$ such that $a_i(\alpha, \mu; t) > 0$, and for each $j = 1, \ldots, N$, $j \neq i$, $a_j(\alpha, \mu; t) < 0$.
2) If for some $i = 1, \ldots, N$ around time $t \in [0, T]$ it is the case that:
$$a_i(\alpha,\mu;t^-) > 0 \text{ and } a_i(\alpha,\mu;t^+) < 0,$$
then there is a single index j, $j = 1, \ldots, N$, $j \neq i$ such that
$$a_j(\alpha,\mu;t^-) < 0 \text{ and } a_j(\alpha,\mu;t^+) > 0.$$
3) For almost all $t \in [0, T]$ and for all $i = 1, \ldots, N$, $\text{sign}(a_i(\alpha, \mu; t)) = \text{sign}(a_i(\mathcal{P}(\alpha, \mu); t))$.

The times for which property 2 occur are the switching times and are the discontinuity points of $Q(\alpha, \mu)$. These times are also the times for which properties 1 and 3 do not hold. Property 1 ensures $Q(\alpha, \mu; t)$ is an element of E^N for almost all time, which is necessary for $Q(\alpha, \mu)$ to be an element of Ω. Property 2 dictates a mode transition at time t, where i is the previous mode and j is the next mode. Property 3 is needed for $\mathcal{P}(\alpha, \mu)$ to be a projection. We refer to the following assumption when a satisfies the above three properties.

Assumption 2: The signal $a(\alpha, \mu)$, which defines the mapping $\mathcal{P}$, Eq.(3), by way of the defined Q, Eq.(4), satisfies the above three properties

Under Assumptions 1 and 2, we see that the mapping $\mathcal{P}$ is a projection.

Lemma 1: Assuming Assumptions 1 and 2, the mapping $\mathcal{P}$, defined through Eq.(3) by Q, which in turn is defined through Eq.(4) by a, is a projection from $\mathcal{R}$ to $\mathcal{S}$.

Proof: To prove $\mathcal{P}$ is a projection, we need to show $\mathcal{P} \circ \mathcal{P}(\alpha, \mu) = \mathcal{P}(\alpha, \mu)$ for each $(\alpha, \mu) \in \mathcal{R}$. First note, by Assumption 1, the solution to $\dot{x} = F(x, Q(\alpha, \mu))$ exists.

Second, according to property 1 of Assumption 2, for almost all time, $a(\alpha, \mu; t)$ is greater than zero for one index and less than zero for all other indexes. From Eq.(4), $Q_i(\alpha, \mu; t) = 1$ if $a_i(\alpha, \mu; t) > 0$ and $Q_i(\alpha, \mu; t) = 0$ if $a_i(\alpha, \mu; t) < 0$. Thus, $Q(\alpha, \mu; t) \in E^N$ and $Q(\alpha, \mu) \in \Omega$.

Third, due to property 3 of Assumption 2, $Q_i(\alpha, \mu; t) = Q_i(\mathcal{P}(\alpha, \mu); t)$ and thus $\mathcal{P}(\alpha, \mu) = \mathcal{P}(\mathcal{P}(\alpha, \mu))$. ■

Example-Max Projection: For $N = 2$, an example of a signal, labeled a_{max}, that satisfies Assumption 2 is

$$a_{max}(\alpha,\mu;t) = \begin{bmatrix} \mu_1(t) - \mu_2(t) \\ \mu_2(t) - \mu_1(t) \end{bmatrix}.$$

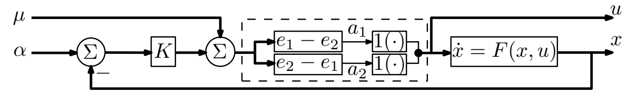

Fig. 1. Block diagram of the $N = 2$ feedback projection. The dashed block converts the continuous signal to a switching control signal—i.e. an element of Ω.

We label the associated reproducing and projection mappings using Q_{max} and $\mathcal{P}_{max}$.

Consider the properties of a for the signal defined by a_{max}. Property 1 is satisfied as long as $\mu_1(t) \neq \mu_2(t)$ for any time interval.[1] We consider the case when $\mu_1(t) = \mu_2(t)$ for a time interval as a degeneracy. As for property 2, it is clearly satisfied for $a_{1,max}(\alpha, \mu) = -a_{2,max}(\alpha, \mu)$. Property 3 is also satisfied for if $\mu_1(t) > \mu_2(t)$ (alt. $<$), then $Q_{max}(\alpha, \mu; t) = [1, 0]^T$ ($[0, 1]^T$) and thus $Q_{1,max}(\alpha, \mu; t) > Q_{2,max}(\alpha, \mu; t)$ ($<$). If Assumption 1 holds, then, by Lemma 1, the mapping $\mathcal{P}_{max}$, given by the signal a_{max}, is a projection.

Example-Feedback Projection: The max projection neglects the unconstrained state. This second example projects to the set of feasible switched system trajectories according to the error of the projected state with the unconstrained state. The operator is designed from the nonlinear trajectory tracking projection operator from [7], [8], which is a continuous linear feedback controller with a feed forward term. We alter the nonlinear projection operator by including a block that takes as input the continuous control signal and outputs the switching control signal. Refer to Fig.(1) for a block diagram. Using the label, K to differentiate between different projections given by different feedback gains, consider the mapping $\mathcal{P}_K$ given by

$$Q_K(\alpha,\mu;t) = \begin{bmatrix} 1\Big[(e_1 - e_2)^T\Big(\mu(t) + K(t)(\alpha(t) - x(t))\Big)\Big] \\ 1\Big[(e_2 - e_1)^T\Big(\mu(t) + K(t)(\alpha(t) - x(t))\Big)\Big] \end{bmatrix},$$

where

$$\dot{x}(t) = F(x(t), Q_K(\alpha,\mu;t)), \quad x(0) = x_0.$$

Here, $a_{1,K} = (e_1 - e_2)^T(\mu(t) + K(t)(\alpha(t) - x(t)))$ and $a_{2,K} = -a_{1,K}$. As [8] suggests for the nonlinear trajectory tracking projection operator, the gain, $K(t) \in \mathbb{R}^{N \times n}$ may be calculated by solving a finite horizon linear regulator problem about (α, μ). Notice that the max projection is the feedback projection with gain $K(t) = 0(t)$—i.e. $Q_{max} = Q_0$. However, when $K(t)$ is non-zero, the value of $Q_K(\alpha, \mu; t)$ depends on $x(t)$, which is variable on all values of (α, μ) prior to time t through solving $\dot{x}(t) = F(x(t), Q_K(\alpha, \mu; t))$.

[1] Suppose $\mu_1(t) = \mu_2(t)$ over some time interval. Certainly, over this time interval, $1(\mu_1(t) - \mu_2(t)) = 1(0(t))$ for calculating $Q_{max}(\alpha, \mu)$, Eq.(4), has meaning. However, in a later section, we will calculate the derivative of the general $Q(\alpha, \mu)$ in a distributional sense when such a calculation is valid. According to [5], for specific a_i the derivative depends on the Dirac delta function where $\delta(\mu_1(t) - \mu_2(t))$ must have meaning. However, $\delta(\mu_1(t) - \mu_2(t)) = \delta(0(t))$ does not have meaning.

The feedback projection commonly encounters Zeno behavior or chattering. To see this, set $\overline{u}(t) = \mu(t) + K(t)(\alpha(t) - x(t))$ and notice $\dot{\overline{u}}(t) = \dot{\mu}(t) + \dot{K}(t)(\alpha(t) - x(t)) + K(t)(\dot{\alpha}(t) - \dot{x}(t))$ where $\dot{x}(t) = f_1(x(t))$ if $\overline{u}_1(t) - \overline{u}_2(t) > 0$ and $\dot{x}(t) = f_2(x(t))$ if $\overline{u}_2(t) - \overline{u}_1(t) > 0$. Now, in order for a mode transition to occur at time t, $\overline{u}_1(t)$ and $\overline{u}_2(t)$ are such that $\|\overline{u}_1(t) - \overline{u}_2(t)\| < \epsilon$, for some ϵ, $1 >> \epsilon > 0$. At this time, Zeno behavior is encountered when $\dot{\overline{u}}_1(t) - \dot{\overline{u}}_2(t)|_{\dot{x}(t)\to f_1(x(t))} < 0$ and $\dot{\overline{u}}_2(t) - \dot{\overline{u}}_1(t)|_{\dot{x}(t)\to f_2(x(t))} < 0$. Certainly, there are f_1 and f_2 for which Zeno behavior cannot occur. Another possibility for dealing with Zeno behavior is to include a hysteresis term so that $\mathcal{P}_K$ only projects to switched system trajectories with an average dwell time [9]. Such analysis is outside the scope of this paper, but this gives an interpretation to average dwell time analysis that parallels classical optimal control.

The mapping $\mathcal{P}_K$ is not a projection unless the mapping $\mathcal{P}_K(\alpha, \mu)$ exists for all $(\alpha, \mu) \in \mathcal{R}$—e.g. the solution does not chatter. However, we can at least show that if $(x, u) \in \mathcal{S}$, then $\mathcal{P}_K(x, u) = (x, u)$.

Lemma 2: If $(x, u) \in \mathcal{S}$, then $\mathcal{P}_K(x, u) = (x, u)$.

Proof: Let $(\overline{x}, \overline{u}) = \mathcal{P}_K(x, u)$—i.e. $\dot{\overline{x}} = F(\overline{x}, \overline{u})$ and $\overline{u} = Q_k(x, u)$. Note, since $(x, u) \in \mathcal{S}$, x and u relate according to $\dot{x} = F(x, u)$. Recall $F(\cdot, \cdot)$ is at least $\mathcal{C}^1$ on $\mathcal{X}$ and as such, x and $\overline{x}$ exist and are both uniquely determined by u and $\overline{u}$ respectively. Notice, at any time $t \in [0, T]$, if $\overline{x}(t) = x(t)$, then $K(t)(\overline{x}(t) - x(t)) = 0$ and we see that

$$\overline{u}(t) = Q_k(x, u, t) = \begin{bmatrix} 1(u_1(t) - u_2(t)) \\ 1(u_2(t) - u_1(t)) \end{bmatrix} = u(t). \quad (5)$$

Initially, $x(0) = \overline{x}(0) = x_0$, so it is also the case that $\overline{u}(0) = u(0)$. Thus, initially as we integrate forward, $\overline{x}(t) = x(t)$. Due to uniqueness of the solution, in order for $\overline{x} \neq x$, there must be some time $t_1 \in [0, T]$ where $\overline{x}(t_1) = x(t_1)$ but $\overline{u}(t_1) \neq u(t_1)$. However, Eq.(5) keeps $\overline{u}(t)$ equal to $u(t)$ as long as $\overline{x}(t) = x(t)$, so t_1 cannot occur. Therefore, $(\overline{x}, \overline{u}) = (x, u)$ and thus $\mathcal{P}_K(x, u) = (x, u)$. ■

In future work, we will develop $\mathcal{P}_K$ so that it does not map to chattering solutions but retains Lemma 2.

B. Derivative of $Q(\cdot)$

Set $\xi = (\alpha, \mu) \in \mathcal{R}$. According to [5], the derivative of $Q_i(\xi; t) = 1(a_i(\xi; t))$ with respect to ξ depends on the Dirac delta function on the hypersurface $a_i(\xi; t) = 0$. However, in order to use the analysis given in [5], we restrict the admissible definitions of the function a_i. First, we only consider a_i that are dependent on the immediate value of ξ at time t. For this reason, we write $a_i(\xi(t))$ where $a_i : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ so that it is understood that a_i is a finite dimensional function. Second, a_i must be sufficiently smooth such that on $a_i(\xi(t)) = 0$, it is the case that $Da_i(\xi(t)) \neq 0$. Now, $\delta(a_i(\xi(t)))$ may be defined as in [5]:

$$\langle \delta(a_i(\xi(t))), \phi \rangle := \int_{a_i(\xi(t))=0} \phi(\xi(t))\omega$$

where ω of degree $n+N-1$ is the differential form uniquely determined by the hypersurface given by $a_i(\xi(t)) = 0$. Suppose the derivative of $a_i(\xi(t))$ with respect to the j^{th} index of $\xi(t)$ is non-zero—i.e. $D_j a_i(\xi(t)) \neq 0$. Then, the differential form is (see [5])

$$\omega = (-1)^{j-1} \frac{d\xi_1(t), \ldots, d\xi_{j-1}(t), d\xi_{j+1}(t), \ldots, d\xi_{n+N}(t)}{D_j a_i(\xi(t))}.$$

Furthermore, according to [5], the derivative of Q is,

$$DQ_i(\xi(t)) = \delta\Big(a_i(\xi(t))\Big) Da_i(\xi(t)) \quad i = 1, \ldots, N. \quad (6)$$

Example-Max Projection: (cont) The derivative of Q_{max} is

$$\left.\begin{aligned} &D_1 Q_{i,max}(\alpha(t), \mu(t)) = 0, \quad i = 1, 2 \\ &D_2 Q_{1,max}(\alpha(t), \mu(t)) = \delta\Big(\mu_1(t) - \mu_2(t)\Big)[1,\ -1]^T \\ &D_2 Q_{2,max}(\alpha(t), \mu(t)) = \delta\Big(\mu_2(t) - \mu_1(t)\Big)[-1,\ 1]^T \end{aligned}\right\} \text{ and}$$

We see $\delta(\mu_1(t) - \mu_2(t))$ is defined as

$$\begin{aligned} &\langle \delta(\mu_1(t) - \mu_2(t)), \phi(\mu_1(t), \mu_2(t)) \rangle \\ &\qquad = \int_{-\infty}^{\infty} \phi(\mu_2(t), \mu_2(t)) d\mu_2(t) \end{aligned}$$

and because the delta function is even, $\delta(\mu_2(t) - \mu_1(t)) = \delta(\mu_1(t) - \mu_2(t))$.

As for the feedback projection, we currently do not have the tools for analyzing its derivative.

IV. Applying the Pontryagin Maximum Principle to Switched System Optimization Using the Projection Operator

The goal is to find the $(\alpha, \mu) \in \mathcal{R}$ which solves

$$\min_{(\alpha,\mu)\in\mathcal{R}} J(\xi) = \int_0^T \ell(x(\tau), Q(\alpha, \mu; \tau))\, \mathrm{d}\tau$$

constrained to

$$\dot{x} = F(x(t), Q(\alpha, \mu; t)), \quad x(0) = x_0. \quad (7)$$

We require the running cost $\ell(\cdot, \cdot)$ to be continuously differentiable with respect to both arguments. Further, assume the projection given by $Q(\alpha, \mu)$ satisfies Assumptions 1 and 2 and that they hold for each of the results discussed in this section. We quickly discuss the Pontryagin Maximum Principle [13] in general terms so that we can apply it to the switched system optimization problem.

A. The Pontryagin Maximum Principle

Suppose the problem at hand is to find the $u \in \mathcal{D} \subset \mathcal{L}_2[0, T]$ that minimizes the cost

$$J(x, u) = \int_0^T \ell(x(\tau), u(\tau))\, \mathrm{d}\tau$$

where (x, u) is constrained to

$$\dot{x} = F(x, u), \quad x(0) = x_0$$

and $F(x, u)$ and $\ell(x, u)$ are continuously differentiable with respect to x and continuous with respect to u. Pontryagin says [13]:

Theorem 3: In order for the point $u^\star \in \mathcal{D}$ to be optimal, there must exist a continuous $\rho(t) = (\rho_1(t), \dots, \rho_n(t))^T \neq 0$ defined through the Hamiltonian,

$$H := H(\rho, x, u) = \rho^T F(x, u) + \ell(x, u)$$

as $\dot{\rho} = -D_2 H(\rho, x, u)^T$ such that

1) for all $t \in [0, T]$, $H(\rho(t), x(t), u)$ of the variable $u \in D$ attains its minimum at $u = u^\star(t)$:

$$H(\rho(t), x(t), u^\star(t)) = \inf_{u \in \mathcal{D}} H(\rho, x, u)$$

2) and $H(\rho(T), x(T), u^\star(T)) = 0$

Assuming $u^\star$ is an interior point of $\mathcal{D}$, then at $u^\star$, it is necessarily true that

$$D_3 H(\rho(t), x(t), u)|_{u \to u^\star(t)} = 0 \tag{8}$$

With regard to switched systems, suppose $\mathcal{D} = \Omega \subset \mathcal{U}$. Clearly, for every point $u \in \Omega$ and for each $\epsilon > 0$, the ball at point u with radius ϵ, $B_u(\epsilon) \in \mathcal{U}$, is not fully contained in Ω. Therefore, the interior of Ω is the null set and Eq.(8) is not a necessary condition for optimality.

In the following subsection, we use the projection operator from the space $\mathcal{R}$ to this set $\mathcal{S}$ with the understanding that the point $\xi^\star = (\alpha^\star, \mu^\star) \in \mathcal{R}$ of the unconstrained problem—i.e. the solution to $\min_{\xi \in \mathcal{R}} J(\mathcal{P}(\xi))$—is in the interior of $\mathcal{R}$ and thus Eq.(8) is a necessary condition for optimality.

B. Applying PMP to Switched Systems

For brevity, label $\xi = (\alpha, \mu)$. Recall, the goal is to find the $\xi \in \mathcal{R}$ which solves the optimization problem:

$$\begin{cases} \min\limits_{\xi \in \mathcal{R}} J(\xi) = \int_0^T \ell\Big(x(\tau), Q(\xi(\tau))\Big)\, \mathrm{d}\tau \\ \text{constrained to: } \dot{x} = F(x(t), Q(\xi(t))), \quad x(0) = x_0. \end{cases} \tag{9}$$

Here, we restrict $Q_i(\xi; t) = 1(a_i(\xi; t))$ to the mappings with functions $a_i(\xi; t)$ considered in Section III-B since under those mappings, we can express the derivative of $Q(\xi(t))$. The following is the necessary condition for optimality of this optimization problem.

Lemma 4: In order for the point $\xi^\star \in \mathcal{R}$ with projected schedule $(\Sigma^\star, \mathcal{T}^\star, M^\star)$ given by the mode sequence $\Sigma^\star = \{\sigma_1^\star, \dots, \sigma_{M^\star}^\star\}$ and switching times $\mathcal{T}^\star = \{T_1^\star, \dots, T_{M^\star-1}^\star\}$ so that

$$Q(\xi^\star(t)) = e_{\sigma_i^\star}, \quad T_{i-1}^\star < t < T_i^\star, \quad i = 1, \dots, M^\star \tag{10}$$

to be the optimum of problem Eq.(9) there must exist a continuous $\rho(t)$ given by

$$\begin{aligned} \dot{\rho}(t) = -D f_{\sigma_i^\star}(x(t))^T \rho(t) - D_1 \ell(x(t), Q(\xi^\star(t)))^T, \\ T_{i-1}^\star < t < T_i^\star, \quad \rho(T) = 0, \end{aligned} \tag{11}$$

such that for $i = 1, \dots, M^\star - 1$,

$$\begin{aligned} 0 = &\Big(\rho^T(T_i) f_{\sigma_i^\star}(x(T_i^\star)) + D_2 \ell\Big(x(T_i^\star), Q(\xi^\star(T_i^\star))\Big)_{\sigma_i^\star}\Big) \\ &\cdot \delta\Big(a_{\sigma_i^\star}(\xi^\star(T_i^\star))\Big) D a_{\sigma_i^\star}(\xi^\star(T_i^\star)) \\ &+ \Big(\rho^T(T_i^\star) f_{\sigma_{i+1}^\star}(x(T_i^\star)) + D_2 \ell\Big(x(T_i^\star), Q(\xi^\star(T_i^\star))\Big)_{\sigma_{i+1}^\star}\Big) \\ &\cdot \delta\Big(a_{\sigma_{i+1}^\star}(\xi^\star(T_i^\star))\Big) D a_{\sigma_{i+1}^\star}(\xi^\star(T_i^\star)). \end{aligned} \tag{12}$$

Proof: The proof follows from the Pontryagin maximum principle, Theorem 3. The Hamiltonian is $H(\rho, x, \xi) = \rho^T F(x, Q(\xi)) + \ell(x, Q(\xi))$ where

$$\begin{aligned} \dot{\rho} &= -D_2 H(\rho(t), x(t), \xi(t))^T \\ &= -D_1 F(x(t), Q(\xi(t)))^T \rho(t) - D_1 \ell(x(t), Q(\xi(t)))^T. \end{aligned} \tag{13}$$

Since $\mathcal{R}$ is open and $F(\cdot, \cdot)$ and $\ell(\cdot, \cdot)$ are differentiable with respect to the second argument, the optimal point $\xi^\star$ must satisfy Eq.(8):

$$\begin{aligned} 0 &= D_3 H(\rho(t), x(t), \xi(t))|_{\xi \to \xi^\star(t)} \\ &= \rho(t)^T D_2 F\Big(x(t), Q(\xi(t))\Big) \circ DQ(\xi(t)) \\ &\quad + D_2 \ell\Big(x(t), Q(\xi(t))\Big) \circ DQ(\xi(t))\Big|_{\xi(t) \to \xi^\star(t)}. \end{aligned} \tag{14}$$

The partial derivative of $F(\cdot, \cdot)$ with respect to its second argument is

$$B(x(t)) := D_2 F\Big(x(t), Q(\xi(t))\Big) = [f_1(x(t)), \dots, f_N(x(t))]$$

and is independent of $Q(\xi)$. Given the derivative of Q from Section III-B, Eq.(14) becomes:

$$\begin{aligned} 0 = &\Big(\rho^T(t) B(x(t)) + D_2 \ell\Big(x(t), Q(\xi(t))\Big)\Big) \\ &\cdot \left.\begin{bmatrix} \delta\Big(a_1(\xi(t))\Big) D a_1(\xi(t)) \\ \vdots \\ \delta\Big(a_N(\xi(t))\Big) D a_N(\xi(t)) \end{bmatrix}\right|_{\xi \to \xi^\star}. \end{aligned} \tag{15}$$

The right side is clearly zero for all time t in which each $a_i(\xi^\star(t))$, $i = 1, \dots, N$ is non-zero. The times when $a_i(\xi^\star(t)) = 0$ for some $i = 1, \dots, N$ are switching times. Let $\mathcal{T}^\star = \{T_1^\star, \dots, T_{M^\star-1}^\star\}$ be the set of times for which an index of $a(\xi^\star(t))$ switches signs. These times are also the discontinuity points of $Q(\xi^\star(t))$. Further, let $\Sigma^\star = \{\sigma_1^\star \dots, \sigma_{M^\star}^\star\}$ be the sequence of indexes i for which $a_i(\xi^\star(t)) > 0$—i.e. for times $t \in (T_{j-1}^\star, T_j^\star)$, $a_{\sigma_j^\star}(\xi^\star(t)) > 0$. Thus, at any switching time $T_i^\star$, $i = 1, \dots, M^\star - 1$, it is the case that both $a_{\sigma_i^\star}(\xi^\star(T_i^\star)) = 0$ and $a_{\sigma_{i+1}^\star}(\xi^\star(T_i^\star)) = 0$. As such, Eq.(15) is reduced to Eq.(12). Furthermore, the adjoint equation, Eq.(13), may be given in its switching schedule form as in (11) completing the proof. ■

In general, Eq.(12) cannot be simplified further because at each of the switching times $T_i^\star$ it is unclear how the hypersurfaces given by $a_{\sigma_i^\star}(\xi(T_i^\star)) = 0$ and $a_{\sigma_{i+1}^\star}(\xi(T_i^\star)) = 0$ relate. However, if the hypersurfaces were given by functions that were so that $a_{\sigma_i^\star}(\xi^\star(T_i^\star)) = -a_{\sigma_{i+1}^\star}(\xi^\star(T_i^\star))$, Eq.(12) may be simplified further. We make the following assumption:

Assumption 3: The projection $\mathcal{P}$, defined through Eq.(3) by Q, which in turn is defined through Eq.(4) by a is so that for each $\xi \in \mathcal{R}$,

$$a_{\sigma_{i+1}}(\xi(T_i)) = -a_{\sigma_i}(\xi(T_i))$$

where T_i and σ_i, $i = 1, \ldots, M-1$ are the switching times and modes of the schedule $(\Sigma, \mathcal{T}, M)$ associated with $Q(\xi)$.

With the addition of Assumption 3 to the projection, we find that the result in Lemma 4 may be further simplified:

Corollary 5: Eq. (12) in Lemma 4 reduces to

$$\begin{aligned} 0 = \rho^T(T_i^\star)\Big(&f_{\sigma_i^\star}(x(T_i^\star)) - f_{\sigma_{i+1}^\star}(x(T_i^\star)\Big) \\ &+D_2\ell\Big(x(T_i^\star), Q(\xi^\star(T_i^\star))\Big)_{\sigma_i^\star} \\ &-D_2\ell\Big(x(T_i^\star), Q(\xi^\star(T_i^\star))\Big)_{\sigma_{i+1}^\star} \end{aligned} \tag{16}$$

if the projection additionally satisfies Assumption 3.

Proof: From Assumption 3,

$$a_{\sigma_{i+1}^\star}(\xi^\star(T_i^\star)) = -a_{\sigma_i^\star}(\xi^\star(T_i^\star)).$$

Since the delta function is even, we note that

$$\delta(a_{\sigma_{i+1}^\star}(\xi^\star(T_i^\star))) = \delta(-a_{\sigma_i^\star}(\xi^\star(T_i^\star))) = \delta(a_{\sigma_i^\star}(\xi^\star(T_i^\star)))$$

for $i = 1, \ldots, M^\star - 1$. Furthermore,

$$Da_{\sigma_{i+1}^\star}(\xi^\star(T_i^\star)) = -Da_{\sigma_i^\star}(\xi^\star(T_i^\star)).$$

As such, Eq.(16) becomes

$$\begin{aligned} 0 = \Big[&\Big(\rho^T(T_i^\star)f_{\sigma_i^\star}(x(T_i^\star)) + D_2\ell\Big(x(T_i^\star), Q(\xi^\star(T_i^\star))\Big)_{\sigma_i^\star}\Big) \\ &-\Big(\rho^T(T_i^\star)f_{\sigma_{i+1}^\star}(x(T_i^\star)) + D_2\ell\Big(x(T_i^\star), Q(\xi^\star(T_i^\star))\Big)_{\sigma_{i+1}^\star}\Big)\Big] \\ &\cdot\delta\Big(a_{\sigma_i^\star}(\xi^\star(T_i^\star))\Big)Da_{\sigma_i^\star}(\xi^\star(T_i^\star)) \end{aligned}$$

In order for the right hand side to be zero, everything multiplying the delta function must be zero. ■

Consider applying Corollary 5 when the projection is the max projection.

Example-Max Projection: (cont) Since $a_{1,max} = -a_{2,max}$, Assumption 3 holds for the max projection. Consider the problem Eq.(9) where the projection is the max projection—i.e. $Q = Q_{max}$. Suppose the running cost, $\ell(x(t), Q(\xi(t)))$ only depends on the state—i.e. $D_2\ell(x(t), Q(\xi(t))) = 0(t)$, which is the case for switching time optimization problems [3], [4], [10], [17]. Applying Corollary 5 we find that the optimal point $\xi^\star$ must associate with an optimal schedule, $(\Sigma^\star, \mathcal{T}^\star, M^\star)$ so that

$$0 = \rho^T(T_i^\star)\Big(f_{\sigma_i^\star}(x(T_i^\star)) - f_{\sigma_{i+1}^\star}(x(T_i^\star)\Big). \tag{17}$$

and ρ is given by Eq.(11).

A couple of remarks: First, this condition on the optimal schedule is equivalent to the hybrid system necessary condition for optimality given in [14], [15] for the optimization problem in this example. Second, the right side of Eq.(17) is the gradient of the cost with respect to the switching times—i.e. $DJ(\mathcal{T})$—as given from the switching time optimization literature [3], [4], [10], [17]. Thus, *any switching time optimal schedule,* $(\Sigma^\star, \mathcal{T}^\star)$ *is a local optimum of the switched system optimization problem, Eq.(9) under the max projection*. More specifically, suppose a switching time optimal schedule has associated control signal and state, $(x^\star, u^\star) \in \mathcal{S}$. Then, any point $(\alpha, \mu) \in \mathcal{P}^{-1}(x^\star, u^\star)$ is a local solution to Eq.(9).

V. Conclusion

This paper introduces projection-based methods for switched system optimization where the mode sequence is a variable. A general form of projection operator is considered, as well as two candidate projections. One is dependent only on the immediate value of the switching control. The other is variable on the projected state error and can be modeled as a feedback control loop. The primary result of the paper shows using the Pontryagin maximum principle that the necessary condition of optimality for the unconstrained problem is equivalent to the necessary condition from switching time optimization. Future work consists of projection-based numerical algorithms for switched system optimization.

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