Variational Nonsmooth Mechanics via a Projected Hamilton's Principle

David Pekarek and Todd D. Murphey

Abstract— **This paper presents a projection-based variational formulation of nonsmooth mechanics. In contrast to existing approaches, nonsmooth behavior is captured through the use of a projection mapping on the configuration space. After clearly defining a projected Hamilton's principle, our main focus is on the existence of admissible projection mappings and variational solutions. This lays the foundation necessary for a variety of useful future developments, which includes optimization techniques, stochastic nonsmooth mechanical system models, and integrator symplecticity proofs.**

I. INTRODUCTION

There are many mechanical systems that provide utility through interaction with their environment and other systems. In doing so, these systems necessarily exhibit behaviors involving collisions and contact between surfaces. The dynamics describing these behaviors will include body velocities, accelerations, and forces that may be nonsmooth or even discontinuous. In the task of modeling these systems, the dominant approach is the generalization of Newton's law to include measure-valued forces using the theory of measure differential inclusions [2]. An alternative to this approach is the extension of variational principles from smooth classical mechanics to include nonsmooth trajectories [19], [11]. Variational formulations give insight into the nonsmooth mechanics and associated conservation laws, but typically involve difficulties when proving the existence and uniqueness of solutions.

In this work, we present a projection-based variational formulation of nonsmooth mechanics, which we refer to as the Projected Hamilton's Principle (PHP). Rather than working with a path space of nonsmooth trajectories, the PHP includes singularities through the use of a nonsmooth projection mapping. Our approach is motivated by the benefits obtained when using projection mappings in variational formulations of optimal control [9], [3]. This paper will demonstrate some benefits already discovered for the PHP. For one, the approach is analogous to the classical treatment of autonomous smooth systems [13]. Specifically, energy conservation arises as a redundant consequence of the PHP and the Hamilton's principle for smooth systems, whereas it is a necessary stationarity condition in the variational principles of [19], [11]. Also, by examining the PHP in discrete time, we have been able to show the variational nature of an existing simulation technique.

D. Pekarek is a Postdoctoral Research Fellow in Mechanical Engineering at Northwestern University, Evanston, IL, USA `d-pekarek@northwestern.edu`

T. D. Murphey is an Assistant Professor in Mechanical Engineering at the McCormick School of Engineering at Northwestern University, Evanston, IL, USA `t-murphey@northwestern.edu`

Beyond these advantages, we anticipate further demonstrable benefits to arise in future applications of the PHP. The usage of smooth variations will simplify the formulation of nonsmooth optimal controls, as it has for switching system optimal controls in [3], and stochastic nonsmooth dynamics, as it has for stochastic dynamics on Lie groups in [4]. Also, the link between discrete variational principles and discrete time symplecticity is well known [14], [8], and thus developing the PHP in discrete time may aid in proving various impact integration methods are symplectic.

To lay the foundation for future developments of the PHP, the main focus of this work is on the existence of its stationary solutions. The existence and uniqueness of solutions for nonsmooth mechanical systems are complex issues, with many of the results requiring the use of measure differential inclusions [12], [18]. The issue is further complicated in our approach, as we must address the existence of both admissible projection mappings and admissible solutions. By restricting to a relevant, structured class of mechanical systems, we are able to analytically derive an admissible projection mapping and link the PHP to existing results.

The structure of this paper is as follows. In Section II, we will review the existing nonsmooth variational principles of [11], [6], and in contrast present the PHP. In Section III, we restrict ourselves to a specific class of Lagrangian systems and explicitly define, local around the impact configuration, a candidate projection mapping for the PHP. In Section IV, we analytically validate that our candidate projection maps to the feasible space, and comment on the existence of variational solutions. In Section V, we discuss the discretization of the PHP and its relation to existing simulation techniques.

II. VARIATIONAL PRINCIPLES FOR NONSMOOTH MECHANICS

In this section, we discuss nonsmooth Lagrangian mechanics for a system undergoing elastic impacts due to the presence of a unilateral constraint. Initially, we review the variational approach seen in [11], [6], which utilizes the notion of a nonsmooth path space. Having provided this context, we present the approach according to the PHP. We utilize a smooth path space, relying on a projection mapping to account for nonsmooth behaviors.

A. Hamilton's Principle with a Nonsmooth Path Space

To begin our discussion of nonsmooth mechanics, we establish the following system model for the remainder of the paper. Consider a mechanical system with configuration space Q (assumed to be an n-dimensional smooth manifold with local coordinates q) and a Lagrangian $L : TQ \to \mathbb{R}$. We will treat this system in the presence of a one-dimensional, holonomic, unilateral constraint defined by a smooth, analytic function $\phi : Q \to \mathbb{R}$ such that the feasible space of the system is $C = \{q \in Q \,|\, \phi(q) \geq 0\}$. We assume C is a submanifold with boundary in Q. Furthermore, we assume that 0 is a regular point of ϕ such that the boundary of C, $\partial C = \phi^{-1}(0)$, is a submanifold of codimension 1 in Q. Physically, ∂C is the set of contact configurations.

The derivation of the nonsmooth impact mechanics of the Lagrangian system above, using the approaches of [11], [6], is as follows. To apply a space-time formulation of Hamilton's principle, we use the following path space of nonsmooth curves

$$\mathscr{M}_{ns} = \mathscr{T} \times \mathscr{Q},$$

where,

$$\begin{aligned} &\mathscr{T} = \{t_i \in (0,1)\}, \\ &\mathscr{Q} = \{q(t) : [0,1] \to C \mid q(t) \text{ is } C^0, \text{ piecewise } C^2, \\ &\qquad \exists \text{ one singularity in } q(t) \text{ at } t_i,\, q(t_i) \in \partial C\}, \end{aligned}$$

where $t_i \in (0,1)$ signifies the time of impact. Moving forward, we will refer to an element of this path space with the shorthand $\bar{q} := (t_i, q(t)) \in \mathscr{M}_{ns}$. On $\mathscr{M}_{ns}$, we define the action $\mathfrak{G}_{ns} : \mathscr{M}_{ns} \to \mathbb{R}$ as

$$\mathfrak{G}_{ns}(\bar{q}) = \int_0^1 L\big(q(t), \dot{q}(t)\big)\, dt. \tag{1}$$

We now examine the first variation of $\mathfrak{G}_{ns}$. As in [13], this is done with variations $\delta\bar{q} \in T_{\bar{q}}\mathscr{M}_{ns}$ defined as

$$\delta\bar{q} := \frac{d}{d\varepsilon}\bar{q}^{\varepsilon}\big|_{\varepsilon=0}, \tag{2}$$

where $\bar{q}^{\varepsilon}$ belongs to a family of curves in $\mathscr{M}_{ns}$ with smooth dependence on ε and $\bar{q}^0 = \bar{q}$. We will also make use of this definition componentwise, with $\delta\bar{q} = (\delta t_i, \delta q(t)) = (\frac{d}{d\varepsilon}t_i^{\varepsilon}|_{\varepsilon=0}, \frac{d}{d\varepsilon}q^{\varepsilon}(t)|_{\varepsilon=0})$. We can now calculate

$$\begin{aligned} \mathbf{d}\mathfrak{G}_{ns}(\bar{q}) \cdot \delta\bar{q} = &\int_0^{t_i} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q(t)\, dt \\ &+ \int_{t_i}^1 \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q(t)\, dt \\ &+ \left[\frac{\partial L}{\partial \dot{q}} \cdot \delta q(t)\right]_0^{t_i^-} + \left[\frac{\partial L}{\partial \dot{q}} \cdot \delta q(t)\right]_{t_i^+}^1 \\ &+ \left[\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right]_{t_i^-}^{t_i^+} \cdot \delta t_i. \end{aligned}$$

Hamilton's principle of stationary action requires that trajectories satisfy $\mathbf{d}\mathfrak{G}_{ns}(\bar{q}) \cdot \delta\bar{q} = 0$ for all variations $\delta\bar{q}$ with $\delta q(0) = \delta q(1) = 0$. This is stated equivalently as

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \tag{3}$$

for all $t \in [0,1] \backslash t_i$,

$$\left[\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right]_{t_i^-}^{t_i^+} = 0, \tag{4}$$

and

$$\left[-\frac{\partial L}{\partial \dot{q}}\right]_{t_i^-}^{t_i^+} \cdot \delta q(t_i) = 0, \tag{5}$$

for all variations $\delta q(t_i) \in T\partial C$. It is crucial to recognize the role of the variation $\delta q(t_i)$ in equation (5). That $\delta q(t_i) \in T\partial C$ rather than $\delta q(t_i) \in TQ$ is an immediate consequence of the constraint $q(t_i) \in \partial C$ imparted by the path space, and it reduces the dimension of equation (5) to $n-1$. Only with the addition of equation (4) do we have a deterministic set of equations of dimension n.

Qualitatively, equation (3) indicates the system obeys the standard Euler-Lagrange equations everywhere away from the impact time, t_i. At the time of impact, equations (4) and (5) imply conservation of energy and conservation of momentum tangent to the impact surface, respectively. Unsurprisingly, these are the standard conditions describing an elastic impact.

B. A Projected Hamilton's Principle

We now consider an alternative to the approach in the prior subsection. Our goal is to, if possible, derive the correct nonsmooth dynamics for elastic collisions without the use of nonsmooth paths and the space $\mathscr{M}_{ns}$. Rather, we use a smooth path space and a projection $P : Q \to C$. With these items we attempt to construct a variational principle with projected solutions satisfying equations (3), (4), and (5).

We begin with the same path space as in the Hamilton's principle for smooth dynamics:

$$\mathscr{M}_s = \{z(t) : [0,1] \to Q \mid z(t) \text{ is } C^2\}.$$

We have made use of the variable z for paths in $\mathscr{M}_s$ to distinguish them from paths $q \in \mathscr{Q}$. Moving forward, it is crucial to keep in mind that paths $q(t)$ are restricted to C, whereas paths $z(t)$ lie in the overlying Q (maybe in part in its inadmissible portion, $Q \backslash C$). Since there are no singularities permitted with $\mathscr{M}_s$, we will introduce nonsmoothness into Hamilton's principle through the use of a projection $P : Q \to C$. Specifically, we work within the family of projection mappings

$$\begin{aligned} \mathscr{P} = \{&P : Q \to C \mid P(P(z)) = P(z), P \text{ is } C^0 \text{ on } Q, \\ &P \text{ is } C^2 \text{ on } Q \backslash \partial C, P|_C \text{ is an isomorphism}, \\ &P|_{Q \backslash C} \text{ is an isomorphism}\}. \end{aligned}$$

Using $P \in \mathscr{P}$, consider the action $\mathfrak{G}_p : \mathscr{M}_s \to \mathbb{R}$ defined as

$$\mathfrak{G}_p(z(t)) = \int_0^1 L\big(P(z(t)), P'(z(t))\dot{z}(t)\big)\, dt, \tag{6}$$

where P' signifies the Jacobian of P. It should be apparent that this action is equivalent to that in equation (1) under the substitution $P\big(z(t)\big) = q(t)$. To make a direct comparison with previous results, when taking variations of $\mathfrak{G}_p$ we will assume $z(t)$ crosses ∂C during one isolated instance at time t_i. Extending the following analysis to multiple crossings is straightforward.[1] Defining $\delta z(t) \in T_{z(t)}\mathscr{M}_s$ in the same manner as $\delta \bar{q}$ in equation (2), we calculate

$$\begin{aligned}\mathbf{d}\mathfrak{G}_p(z(t)) \cdot \delta z(t) = &\int_0^{t_i} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] P' \delta z(t)\, dt \\ &+ \int_{t_i}^{1} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] P' \delta z(t)\, dt \\ &+ \left[\frac{\partial L}{\partial \dot{q}} P' \cdot \delta z(t)\right]_0^{t_i^-} + \left[\frac{\partial L}{\partial \dot{q}} P' \cdot \delta z(t)\right]_{t_i^+}^{1}.\end{aligned}$$

If it is not evident, all instances of $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}}$ are evaluated at $(P(z(t)), P'(z(t))\dot{z}(t))$ and all instances of P' are evaluated at $z(t)$.

Again employing Hamilton's principle, we require that trajectories satisfy $\mathbf{d}\mathfrak{G}_p(z(t)) \cdot \delta z(t) = 0$ for all variations $\delta z(t)$. For this case, the stationarity conditions are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \tag{7}$$

for all $t \in [0,1] \backslash t_i$, and

$$\left[-\frac{\partial L}{\partial \dot{q}} P'\right]_{t_i^-}^{t_i^+} = 0. \tag{8}$$

Note that equation (7) appears without the presence of P' only as a result of the invertibility condition in $\mathscr{P}$. That P is invertible implies span($\delta z(t)$) = span($P' \cdot \delta z(t)$) for all $t \neq t_i$ and thus equation (7), or equivalently equation (3), holds. The question remains, do projected solutions $P(z(t))$ satisfy equations (4) and (5) as well? We cannot make any conclusions without additional structure in the definition of P, which we explore in the following section.

III. DEFINING A CANDIDATE PROJECTION

In this section, we further analyze the potential equivalence between the variational solutions in subsections II-A and II-B. First, we continue work with the general case as long as possible, through the use of additional restrictions on the projection mapping P. Eventually, we require more information about the interaction between L and P. Thus, we restrict the class of systems we wish to consider and analytically define a candidate projection P about the impact configuration.

A. Additional Constraints on the Projection

First, we consider conditions for projected solutions $P(z(t))$ to satisfy equation (5). In the following, and for the remainder of the paper, we use the shorthand $z_i = z(t_i)$ for the impact configuration. We note that if

$$P' \cdot \delta z_i = \delta z_i, \tag{9}$$

for all $\delta z_i \in T_{z_i}\partial C$, then solutions of equation (8) will satisfy equation (5) as well. This is not a necessary but sufficient condition on P, and one we will use.

[1]Truthfully, it is straightforward with the previous approach as well, though it requires changes in the definition of $\mathscr{M}_{ns}$.

Secondly, we turn to $P(z(t))$ satisfying equation (4). Consider just the first term, $\frac{\partial L}{\partial \dot{q}}\dot{q}$, and examine that by postmultiplying equation (8) by $\dot{z}(t_i)$ we have

$$\begin{aligned}0 &= 0 \cdot \dot{z}(t_i), \\ &= \left[\frac{\partial L}{\partial \dot{q}} P'\right]_{t_i^-}^{t_i^+} \dot{z}(t_i), \\ &= \left[\frac{\partial L}{\partial \dot{q}} \frac{d}{dt}\big(P(z)\big)\right]_{t_i^-}^{t_i^+},\end{aligned}$$

where we have made use of the property $\dot{z}(t_i^-) = \dot{z}(t_i^+)$. Thus, without any additional conditions on $P \in \mathscr{P}$, we have that solutions of equation (8) conserve $\frac{\partial L}{\partial \dot{q}}\dot{q}$ through the impact. This reduces equation (4) to the equivalent condition

$$\Big[-L\big(P(z), P'(z)\dot{z}\big)\Big]_{t_i^-}^{t_i^+} = 0. \tag{10}$$

Seemingly, we cannot make further determinations about this condition without more information about the Lagrangian, L. Thus, to progress, we assume a specific form of the Lagrangian in the next subsection.

B. Restricting the Class of Lagrangians

In robotics [15], it is common to see a variety of systems (for instance, mechanisms, bipeds, etc.) modeled with Lagrangians of the form

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \tag{11}$$

where $M(q)$ is a symmetric positive definite mass matrix and $V(q)$ is a potential function. We will proceed working with the Lagrangian (11) and the simplifying assumption[2] $Q = \mathbb{R}^n$. We now have $\frac{\partial L}{\partial \dot{q}} = \dot{q}^T M(q)$ and

$$\frac{\partial L}{\partial \dot{q}}\dot{q} - L = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q).$$

As seen in [16], [5], [17], if $M(q)$ is invertible on all of ∂C then the equations (4) and (5) yield the explicit solution[3]

$$\dot{q}(t_i^+) = R(q(t_i))\dot{q}(t_i^-), \tag{12}$$

with

$$R(q(t_i)) = \left(\mathbb{I} - \frac{2}{\nabla\phi^T M^{-1} \nabla\phi} M^{-1} \nabla\phi \nabla\phi^T\right),$$

where all instances of $\nabla\phi$ and M^{-1} are evaluated at the argument $q(t_i)$ and $\mathbb{I}$ signifies the $n \times n$ identity matrix. We will use this solution as a guide to construct P about z_i.

Let us assume that P is the identity on C, a natural choice. Under this assumption $q(t_i) = z(t_i)$ and $\dot{q}(t_i^-) = \dot{z}(t_i^-) = \dot{z}(t_i^+)$, where the final equality follows from the continuity of

[2]This assumption is not required for the impact mechanics that follow, and is primarily used to simplify the definition of P and the end of the subsection. We believe we can extend the given definition of P to general manifolds, but exclude this extension for brevity.

[3]In actuality, (4) and (5) also admit a second solution $\dot{q}(t_i^-) = \dot{q}(t_i^+)$, but we disregard this as it would cause $q(t)$ to exit the feasible space C for $t > t_i$.

$\dot{z}$. Substituting in these relations and $P'(z)\dot{z}$ for $\dot{q}$ in equation (12), we arrive at

$$P'(z(t_i^+))\dot{z}(t_i^+) = R(z_i)\dot{z}(t_i^+),$$

or simply $P'(z(t_i^+)) = R(z_i)$. We note that this definition of P' satisfies condition (9), a consequence of $\nabla\phi^T \cdot \delta z_i = 0$ for all $\delta z_i \in T_{z_i}\partial C$. Also, the combination $P'(z(t_i^-)) = \mathbb{I}$ and $P'(z(t_i^+)) = R(z_i)$ satisfies equation (10) since

$$R(z_i)^T M(z_i) R(z_i) = M(z_i),$$

and V is independent of $\dot{q}$. Thus, we have a Jacobian P' that yields the correct impact dynamics through equation (8).

We must now lift the P' above to a candidate projection P. Remembering that we are working in $Q = \mathbb{R}^n$, one mapping P with the correct Jacobian is

$$P(z) = \begin{cases} z, & z \in C, \\ z - \frac{2\phi(z)}{\nabla\phi^T M^{-1}\nabla\phi} M^{-1}\nabla\phi, & z \in Q\backslash C, \end{cases} \tag{13}$$

where all instances of $\nabla\phi$ and M^{-1} are evaluated at z_i and only z and $\phi(z)$ are evaluated at the argument z. This P produces the correct impact dynamics with C^2 solutions $z(t)$ satisfying the stationarity condition (8). However, we cannot guarantee that this mapping is actually a projection $P \in \mathscr{P}$. Also, in the instance that it is, we would like some guarantee on the existence of solutions $z(t)$. These issues are explored in the following section.

IV. VALIDATION OF THE CANDIDATE PROJECTION

In this section, we examine the properties of the candidate projection P in equation (13). While we cannot generally verify $P \in \mathscr{P}$, we will show analytically that there exists an open neighborhood $N \subset Q$ local to the impact configuration z_i in which P acts as a projection. With this property, we then discuss the existence of solutions under the PHP. We conclude examining an example system, the double pendulum, and computationally verify the existence of N.

A. Locally Validating the Projection about the Impact

Examining the definition of the candidate P in (13), we cannot generally say $P \in \mathscr{P}$ because we have no guarantee that $P : Q \to C$. Further, we cannot verify this property without more information about Q, C, and ϕ. Rather than invoking further restrictions and assumptions to prove this, we provide the following lemma.

Lemma 1: There exists an open neighborhood $N \subset Q\backslash C$ of z_i such that $P : N \cup C \to C$.

Proof: Consider a general $z \in Q\backslash C$, and examine the Taylor expansion

$$\begin{aligned} \phi(z) &= \phi(z_i) + \nabla\phi^T (z - z_i) + O\left(\|z - z_i\|^2\right), \\ &= \nabla\phi^T (z - z_i) + O\left(\|z - z_i\|^2\right), \end{aligned}$$

where $\|\cdot\|$ is the standard Euclidean metric on $\mathbb{R}^n$. We substitute this expansion in the following

$$\begin{aligned} P(z) - z_i &= z - z_i - \frac{2\phi(z)}{\nabla\phi^T M^{-1}\nabla\phi} M^{-1}\nabla\phi, \\ &= R(z_i)(z - z_i) + O\left(\|z - z_i\|^2\right), \\ &= O(\|z - z_i\|), \end{aligned}$$

indicating $\|P(z) - z_i\|^2 = O\left(\|z - z_i\|^2\right)$. With this, we expand $\phi(P(z))$ as

$$\begin{aligned} \phi(P(z)) &= \nabla\phi^T (P(z) - z_i) + O\left(\|P(z) - z_i\|^2\right), \\ &= \nabla\phi^T (P(z) - z_i) + O\left(\|z - z_i\|^2\right), \\ &= \nabla\phi^T R(z_i)(z - z_i) + O\left(\|z - z_i\|^2\right), \\ &= -\nabla\phi^T (z - z_i) + O\left(\|z - z_i\|^2\right), \\ &= -\phi(z) + O\left(\|z - z_i\|^2\right). \end{aligned}$$

Noting that $\nabla\phi$ is nonzero at z_i, there exists a neighborhood N of z_i in which $-\phi(z) = O(\|z - z_i\|)$ and $\mathrm{sgn}(\phi(P(z))) = -\mathrm{sgn}(\phi(z)) > 0$ for all $z \in N$. Therefore $P(N) \to C$, and since P is the identity on C the extension to $P : N \cup C \to C$ is trivial. ■

In the following subsection, we discuss how the existence of N provided by Lemma 1 may provide for the local existence of solutions.

B. Existence of Variational Solutions $z(t)$

Global existence and uniqueness of solutions for nonsmooth mechanical systems are complex issues, and results are often intractable. However, it is mentioned in [6] that for the Lagrangian (11) with quadratic kinetic energy and the boundary ∂C of codimension-one in Q, solutions to equations (3), (4), and (5) exist and are unique locally. Given the complexity introduced by the many-to-one P in our projected Hamilton's principle, we focus our discussion on the existence of solutions and save local uniqueness as the subject of future work. With regards to existence, we have the following results.

While we have been working with P as defined in equation (13), we note here that we really only use that definition for its value of P' at ∂C. There is substantial freedom to smoothly vary P on $Q\backslash C$ away from ∂C, as illustrated in the following lemma.

Lemma 2: Given the following:

- a Lagrangian of the form (11) on $Q = \mathbb{R}^n$,
- a boundary ∂C of codimension-one,
- $P \in \mathscr{P}$,
- for all $\zeta > 0$ there exists $\eta > 0$ such that for any $z \in Q\backslash C$ and $z_i \in \partial C$, $\|z - z_i\| < \eta$ implies $\|P'(z) - R(z_i)\| < \zeta$, where $\|\cdot\|$ applied to elements of $\mathbb{R}^{n\times n}$ is the induced matrix norm of the Euclidean metric,

there exists $z(t) \in Q$ such that $\mathbf{d}\mathfrak{G}_p(z(t)) \cdot \delta z(t) = 0$ and $P(z(t))$ satisfies equations (3), (4), and (5).

Proof: Given the system parameters, we know there exists $q(t)$ satisfying (3), (4), and (5). We construct $z(t)$

explicitly as

$$z(t) = \begin{cases} q(t), & t \in [0, t_i], \\ P|_{Q\backslash C}^{-1}(q(t)), & t \in (t_i, 1]. \end{cases}$$

That this $z(t)$ satisfies equation (7) is trivial. The given condition on $P'(z)$ indicates $P'(z(t_i^+))\dot{z}(t_i^+) = R(z_i)\dot{z}(t_i^+)$, and thus (5) is satisfied as well. Lastly, in subsection II-B it was shown (7) and (5) are equivalent to $\mathbf{d}\mathfrak{G}_p(z(t)) \cdot \delta z(t) = 0$. ■

The above lemma provides a general condition to identify a subset of $\mathscr{P}$ that produces the correct elastic collision dynamics under the PHP. For some unilateral constraints ϕ, the projection P in (13) falls into this subset.

Corollary 3: Given the following:

- a Lagrangian of the form (11) on $Q = \mathbb{R}^n$,
- a planar boundary ∂C defined by the linear constraint $\phi(q) = b^T q$, where $b \in \mathbb{R}^n$,

there exists $z(t) \in Q$ and $P \in \mathscr{P}$ such that $\mathbf{d}\mathfrak{G}_p(z(t)) \cdot \delta z(t) = 0$ and $P(z(t))$ satisfies equations (3), (4), and (5).

Proof: Given the system parameters, we know there exists $q(t)$ satisfying (3), (4), and (5). Hence, there exists $z_i = q(t_i)$ such that

$$R(z_i) = \left(\mathbb{I} - \frac{2}{b^T M^{-1} b} M^{-1} b b^T \right),$$

where M^{-1} is evaluated at z_i. Using the definition of P in (13), we have $P|_{Q\backslash C}(z) = R(z_i)z$. Noting $\phi(P|_{Q\backslash C}(z)) = -\phi(z)$, we have $N = Q\backslash C$. Additionally, $R(z_i)^2 = \mathbb{I}$ indicates $P|_{Q\backslash C}^{-1}(z) = R(z_i)z$. From here, it is straightforward to verify $P \in \mathscr{P}$ and apply Lemma 2. ■

Existence results when using P as in equation (13) with arbitrary ϕ are less certain. Essentially, we cannot determine if $P|_N : N \to P(N)$ is an isomorphism in general. When $P|_N$ is invertible, one can construct a solution $z(t)$ from an existing solution $q(t)$ as in Lemma 2, but only so long as $q(t) \in P(N)$ following the impact. This is not considered to be a major issue, as the system could be evolved further in time by reinitializing with $z(t) = q(t)$.

C. Computational Validation for an Example System

To get some sense of the nature of the neighborhood N, we present the following example. Consider a double pendulum in the plane composed of two point masses, m_1 and m_2, connected in sequence (from the origin) by inertialess rods of respective lengths L_1 and L_2. We henceforth refer to the rods by their lengths. For simplicity,[4] we specify the configuration space as $Q = \mathbb{R}^2$ with coordinates $q = (\theta_1, \theta_2)$, where θ_1 is the angle of L_1 with respect to vertical and θ_2 is the angle of L_2 with respect to L_1. Assuming there exists only potential forces and they are independent of $\dot{q}$, the double pendulum fits the form of (11) with

$$M(q) = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1 L_2 c_2) & m_2(L_2^2 + L_1 L_2 c_2) \\ m_2(L_2^2 + L_1 L_2 c_2) & m_2 L_2^2 \end{bmatrix},$$

[4] While the true configuration space of the double pendulum is the 2-torus, T^2, we work in $\mathbb{R}^2$ to remain true to our prior assumptions.

where we have introduced the shorthand $c_i = \cos\theta_i$.

When introducing impacts into the system, we apply the unilateral constraint

$$\phi(q) = L_1 s_1 + L_2 s_{12} + 1,$$

where $s_i = \sin\theta_i$ and $s_{ij} = \sin(\theta_i + \theta_j)$. Physically, this constrains the horizontal position of m_2 (the end of the double pendulum) to values greater than or equal to -1.

Using $m_1 = m_2 = L_1 = L_2 = 1$, we have computationally sampled the configuration space Q to verify the existence of the subset N for a variety of impact configurations z_i. The results are shown in Figure 1. We see that for each impact condition, there is a blue set N indicating a domain on which a solution $z(t)$ may evolve past a collision and into $Q\backslash C$. The existence of a red area in each plot, composed of z for which $P(z) \notin C$, means we should not expect solutions $z(t)$ to exist on a long time horizon, and reinitializing as discussed in the prior subsection may be required.

V. DISCRETE TIME NONSMOOTH MECHANICS WITH PROJECTIONS

As it is often the case that we cannot determine analytical solutions $z(t)$ satisfying equations (7) and (8), we regularly turn to simulation. With our emphasis on the variational nature of solutions, a natural choice of simulation method is the use of discrete mechanics and variational integrator (VI) theory [14], [8]. VIs are generated with discrete time variational principles and represent a class of symplectic-momentum integration schemes. The work of [6] presents a VI for the nonsmooth mechanics of subsection II-A, although with known energy stability issues [16]. In the following section, we explore a discrete time version of the projected Hamilton's principle in subsection II-B.

A. Variational Integrators via a Discrete Projected Hamilton's Principle

Discrete mechanics specifies that in order to capture smooth dynamics in discrete time we begin by replacing our notion of the state space TQ with $Q \times Q$ and substitute for the continuous trajectory $z(t)$ a discrete path $z_d : \{0, h, \ldots, jh, (j+\alpha)h, (j+1)h, \ldots Nh = 1\} \to Q$, $N \in \mathbb{N}$, where h is a constant timestep. This path is defined such that $z_k := z_d(kh)$ is considered an approximation to $z(kh)$. Notice that we treat the existence of an impact along the path z_d in the same manner as [6]. That is, we assume knowledge of the time interval $[jh, (j+1)h]$, with $j \in \mathbb{N}$ and $j < N$, in which the impact occurs. Then we use $\alpha \in (0,1)$ to parameterize the partial timestep, αh, that precisely identifies the impact time, $(j+\alpha)h$. With this convention, the discrete configuration $z_{j+\alpha} := z_d((j+\alpha)h) \in \partial C$ approximates the continuous time impact configuration $z(t_i)$.

Based on these discretizations, the action integral in Hamilton's principle is approximated on discrete intervals of time using a discrete Lagrangian $L_d : Q \times Q \times \mathbb{R}$ such that

$$L_d(z_k, z_{k+1}, h) \approx \int_{kh}^{(k+1)h} L(z(t), \dot{z}(t))\, dt.$$

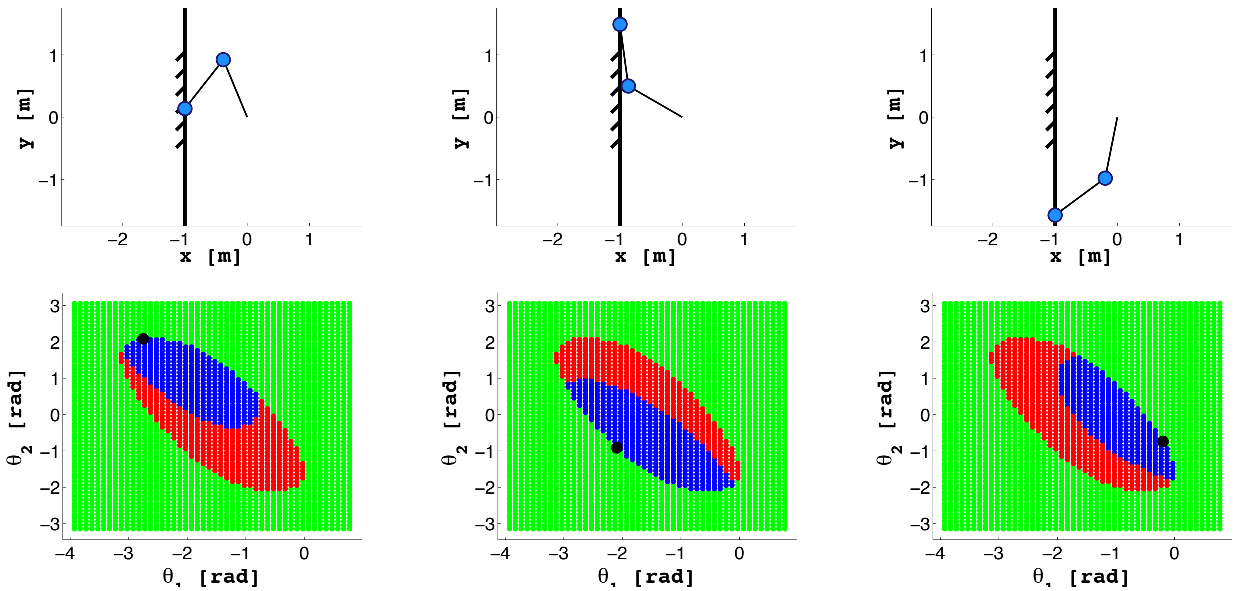

Fig. 1. For a variety of impact configurations (top row), we computationally validate that the mapping P acts as a projection on some subset $N \subset Q\backslash C$. Using samples of the double pendulum's configuration space Q (bottom row), we see the impact configuration z_i (black dot), the feasible space C (green), the neighborhood $N \subset Q\backslash C$ with $P(N) \subset C$ (blue), and the remainder $Q\backslash(C \cup N)$ (red). For this system, N accounts for a sizeable portion of $Q\backslash C$.

In practice, L_d is defined using quadrature rules, with higher order rules resulting in higher order integrators [14]. Regardless of the choice of discrete Lagrangian, one can approximate the action (6) with the discrete action

$$\begin{aligned}\mathfrak{G}_d(z_d) = &\sum_{k=0}^{j-1} L_d\big(P(z_k), P(z_{k+1}), h\big) + L_d\big(P(z_j), P(z_{j+\alpha}), \alpha h\big)\\ &+ L_d\big(P(z_{j+\alpha}), P(z_{j+1}), (1-\alpha)h\big) \qquad (14)\\ &+ \sum_{k=j+1}^{N-1} L_d\big(P(z_k), P(z_{k+1}), h\big).\end{aligned}$$

We now examine the consequences of a discrete projected Hamilton's principle that requires solutions to satisfy $\mathbf{d}\mathfrak{G}_d(z_d)\cdot\delta z_d = 0$ for all variations δz_d with $\delta z_0 = \delta z_N = 0$. Using q_i as shorthand for $P(z_i)$ and D_i to denote the slot derivative, the stationarity conditions for this case are

$$0 = D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h), \qquad (15)$$

for $1 \le k \le N-1$ with $k \ne j, j+1$, and around the impact

$$0 = D_2 L_d(q_{j-1}, q_j, h) + D_1 L_d(q_j, q_{j+\alpha}, \alpha h), \qquad (16)$$

$$\begin{aligned}0 = &D_2 L_d(q_j, q_{j+\alpha}, \alpha h) P'(z^-_{j+\alpha})\\ &+ D_1 L_d(q_{j+\alpha}, q_{j+1}, (1-\alpha)h) P'(z^+_{j+\alpha}), \qquad (17)\end{aligned}$$

$$\begin{aligned}0 = &D_2 L_d(q_{j+\alpha}, q_{j+1}, (1-\alpha)h)\\ &+ D_1 L_d(q_{j+1}, q_{j+2}), h). \qquad (18)\end{aligned}$$

Equation (15) is referred to as the discrete Euler-Lagrange (DEL) equation and in this case, remembering our shorthand $q_i = P(z_i)$, it provides an implicit map from z_{k-1}, z_k to z_{k+1}. In practice, one would integrate solving the DEL equation for z_{k+1} until a crossing of ∂C is indicated by the condition $\mathrm{sgn}(\phi(P(z_{k+1}))) \ne \mathrm{sgn}(\phi(P(z_k)))$. This occurrence would identify k as j, and the existing z_{k+1} must be disregarded. Instead of using that solution, one solves equation (16) in combination with the condition $z_{j+\alpha} \in \partial C$ to determine $z_{j+\alpha}$ and α. Following this, one solves equation (17) for z_{j+1}, equation (18) for z_{j+2}, and returns to using the DEL equations until the next collision is identified.

We now turn our attention to an existing impact integration method, and its relation to the discrete time impact equations (16), (17), and (18). The Collision Verlet Algorithm (CVA) of [10] was designed for integrating Hamiltonian system models of molecular dynamics with hard-core potentials. The discrete time impact equations for the CVA read[5]

$$\Big(q^-_{j+\alpha}, p^-_{j+\alpha}\Big) = \Psi_{\alpha h}\Big(q_j, p_j\Big), \qquad (19)$$

$$\Big(q^+_{j+\alpha}, p^+_{j+\alpha}\Big) = R_{\mathrm{CVA}}\Big(q^-_{j+\alpha}, p^-_{j+\alpha}\Big), \qquad (20)$$

$$\Big(q_{j+1}, p_{j+1}\Big) = \Psi_{(1-\alpha)h}\Big(q^+_{j+\alpha}, p^+_{j+\alpha}\Big), \qquad (21)$$

where p_i represents the momentum[6] of the Hamiltonian system in discrete time, $\Psi_h : T^*Q \to T^*Q$ represents the discrete flow map for the Störmer-Verlet [7] method, and $R_{\mathrm{CVA}} : T^*Q \to T^*Q$ is the reset map

$$R_{\mathrm{CVA}}(q,p) = \big(q, p + \lambda \nabla\phi^T\big),$$

where $\nabla\phi$ is evaluated at q, and $\lambda \in \mathbb{R}$ is a Lagrange multiplier sized appropriately to conserve energy. We have the following result comparing our variational equations with the CVA.

[5]The algorithm in [10] is defined to handle an arbitrary number of collisions per time step. We present a simplified version of the algorithm under a single collision assumption.

[6]Momentum, $p \in T^*_q Q$, is a covector and is thus treated as a row vector in $\mathbb{R}^n$ for the remainder of our calculations.

Lemma 4: Given the following:

- Q, L, ∂C, and P as in Lemma 2,
- M is an global isomorphism on Q,
- the Störmer-Verlet producing discrete Lagrangian

$$L_d(q_k, q_{k+1}, h) = \frac{h}{2}\left(L\left(q_k, \frac{q_{k+1}-q_k}{h}\right) + L\left(q_{k+1}, \frac{q_{k+1}-q_k}{h}\right)\right),$$

the variational discrete impact equations (16), (17), and (18) are equivalent to the CVA equations (19), (20), and (21).

Proof: As we are examining integration methods in both the Lagrangian and Hamiltonian and settings, we facilitate their comparison by introducing continuous and discrete Legendre transformations [14]

$$(q,p) = \left(q, \frac{\partial L}{\partial \dot{q}}\right), \tag{22}$$

$$(q_{k+1}, p_{k+1}) = (q_{k+1}, D_2 L_d(q_k, q_{k+1}, h)), \tag{23}$$

$$(q_k, p_k) = (q_k, -D_1 L_d(q_k, q_{k+1}, h)). \tag{24}$$

The discrete transformations (23) and (24) immediately provide equivalence between the respective pairs of partial timestepping equations (16), (18) and (19), (21). Substituting (11) into (22), we have $p^T = M(q)\dot{q}$ and

$$\begin{aligned} R_{\mathrm{CVA}}(q,p) &= \left(q, \left(M(q)R(q)M^{-1}(q)p^T\right)^T\right), \\ &= \left(q, \left(R^T(q)p^T\right)^T\right), \\ &= (q, pR(q)). \end{aligned}$$

With this simplified form of R_{CVA}, we see that (17) is equivalent to (20) by substitutions of $P'(z^+_{j+\alpha})) = R(z_{j+\alpha})$ and (23), (24) and a postmultiplication by $R(z_{j+\alpha})$. ■

That we can demonstrate the CVA is variational in nature is a significant development. Further analysis of the discrete PHP may provide a proof of the symplecticity of the CVA, which has thus far not been determined.

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

Our variational formulation of nonsmooth mechanics uses smooth trajectories and a nonsmooth projection mapping on the configuration space. With this approach we have produced results analogous to those for autonomous smooth systems, such as the condition of energy conservation as a consequence of, not condition for, stationarity. By restricting the class of systems considered, we have produced preliminary results for the existence of admissible projection mappings and stationary solutions. Examining our projected Hamilton's principle in discrete time, we revealed the variational nature of the CVA for impact integration.

B. Future Works

As discussed at the outset, there is great potential for further applications of our projected Hamilton's principle. Moving forward, we intend to add forcing and controls, eventually producing optimal control generation methods in a manner analogous to [9]. Further, we may introduce stochastic effects in our system model, with the projected Hamilton's principle aiding in the derivation of a version of the Fokker-Planck equations for nonsmooth systems. Working in discrete time, we hope to determine the symplecticity, if it truly exists, of the CVA. Also, we intend to demonstrate the variational nature of other existing integration techniques by using different choices of the projection P.

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