Global Projections for Variational Nonsmooth Mechanics

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***Abstract*— This paper improves the projected Hamilton's principle (PHP) formulation of nonsmooth mechanics. Central to the PHP is the use of a projection mapping, defined on the configuration space, to capture nonsmooth behaviors. To support applications of the PHP with multiple impact times and locations, we define mild topological assumptions under which nonsmooth mechanical systems can be transformed to a prescribed normal form. In normal form coordinates, we provide a globally valid projection for use in the PHP. For systems that do not permit the transformation to normal form, we examine the use of constrained coordinates and incorporate holonomic constraints into the PHP. Lastly, as a preview of future developments of the PHP, we discuss the application of the method on compact manifolds.**

I. INTRODUCTION

In the field of nonsmooth mechanics there exists a number of valid approaches to system modeling and simulation, each with its own unique characteristics. One may use measure differential inclusion formulations [1], [2], [3], which offer some of the most powerful existence and uniqueness results for nonsmooth trajectories. Alternatively, nonsmooth system dynamics may be modeled as linear complementarity problems [4], [5], [6], which offer robustness in computational applications but some costs in model accuracy. Yet another common approach is that of barrier methods [7], [8], which yield energy conservation properties and feasibility guarantees through the regularization of contact impulses into smooth potential forces.

In contrast to all of these approaches, we seek formulations of nonsmooth mechanics that derive impact dynamics as the stationarity conditions of prescribed variational principles. Variational methods have a rich history in the general field of mechanics, giving insight into the geometric structure and conservation laws of mechanical systems [9], and motivating structured models of these systems in discrete time [10], [11]. In fact, the specific use of variational methods for nonsmooth mechanics has been explored prior in [12], [13]. However, the formulation herein, along with the authors' prior work [14], differentiates itself through the use of projection mappings. Rather than searching for stationary solutions in a path space of nonsmooth curves, as is the practice in [12], [13], we formulate a Projected Hamilton's Principle (PHP) that utilizes a smooth path space and captures nonsmooth behaviors with the presence of a differentiable, nonsmooth projection mapping.

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We find the general structure of the PHP appealing for the following reasons. For one, the presence of smooth variations in an autonomous setting parallels the well-understood, classical treatment of smooth mechanical systems. Beyond this, smooth variations play a central role in the formulations of stochastic dynamics on Lie groups in [15], and thus will facilitate the development of stochastic dynamic models of nonsmooth mechanical systems. Also, the use of projection mappings has been beneficial in the optimal control techniques of [16], [17], and we anticipate their presence in the PHP will enable powerful methods for the optimal control of nonsmooth systems. Lastly, discrete time representations of the PHP provide a tool for the analysis of simulation methods [14], with which one can identify the discrete variational structure and conservation laws of a given timestepping scheme.

With this in mind, the contributions of this work are largely foundational. That is, we identify sufficient conditions and, when possible, design general projection mappings by which the PHP's stationarity conditions correctly represent impact dynamics. To differentiate these contributions from those of the prior work [14], we present the following simple example. Consider a planar particle mass with coordinates (x, y) subject to a unilateral constraint $\phi_u \geq 0$, where

$$\phi(x, y) = y + 2\sin x.$$

Both [14] and the work herein present projection mapping designs by which the PHP will properly represent this particle's impact dynamics. The difference in the results associated with the respective projections is highlighted in Figure 1. The projection map of [14], the qualitative behavior of which is featured in the leftmost plot, utilizes knowledge of the impact configuration in its definition and is restricted to a domain of validity local to that configuration. Essentially, this design is unable to facilitate trajectories with multiple collisions or trajectories that depart extensively from the point of collision. In contrast, the projection design in this work solves both of these issues by identifying sufficient conditions for a coordinate transformation that linearizes the constraint surface (middle plot). In these coordinates, we design a map that globally projects all infeasible configurations to the feasible space, and correctly captures impact dynamics regardless of the number and location of impacts. Since the specified coordinate transformation and the projection are diffeomorphisms, these results can be transformed back to the given original coordinate system if desired (right plot).

The structure of this paper is as follows. In Section II, we will review the existing nonsmooth variational principles of [12], [13] and the PHP results of [14]. In Section III,

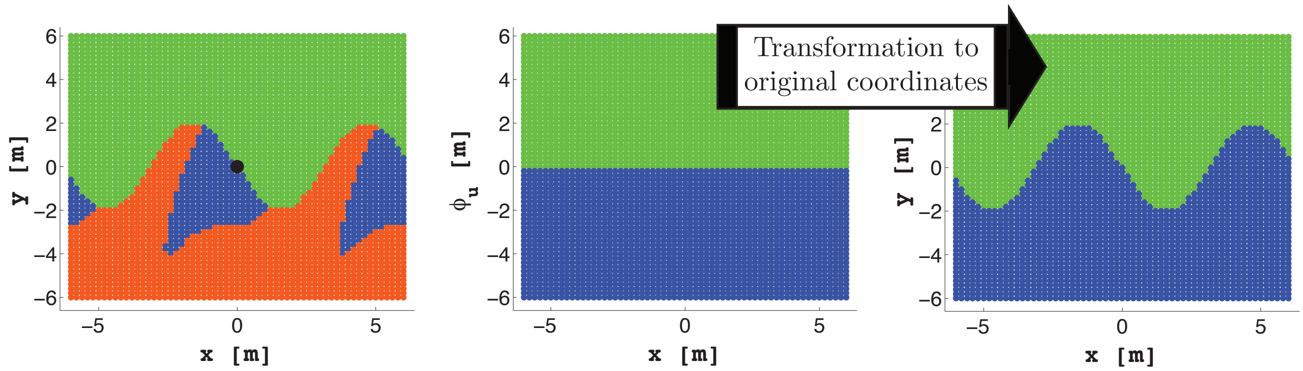

Fig. 1. A comparison of PHP projection mappings via computational sampling of the configuration space of the constrained planar particle. Left: The projection map of [14] is defined in terms of a given impact configuration (black dot), and only projects a region local to the impact (blue) onto the feasible space (green). A large subset of infeasible configurations (red) is not successfully projected to the feasible space. Middle: By treating the system in normal form coordinates, the projection mapping of Section III globally projects infeasible configurations to the feasible space, regardless of the impact configuration (absence of black dot). Right: Transformation of the projection back to the system's original (x, y) coordinates does not change its global application.

we define a normal form for nonsmooth Lagrangian systems and explicitly define a global projection mapping for systems that permit that form. For systems that do not permit a transformation to normal form, we discuss the potential use of constrained coordinates in Section IV and compact configuration manifolds in Section V.

II. VARIATIONAL NONSMOOTH MECHANICS

In this section, we review several results when deriving nonsmooth Lagrangian mechanics from variational principles. Initially, we present the classical approach seen in [12], [13], which utilizes the notion of a nonsmooth path space. Next, we present the projected smooth path space approach of [14]. For each approach, we focus on the governing dynamics that result from the respective variational principles. For a detailed derivation of these dynamics as stationarity conditions, refer to the aforementioned references.

A. Nonsmooth Mechanics via a Nonsmooth Path Space

To begin our discussion of nonsmooth mechanics, we establish the following system model (the same used in [14]) for the remainder of the paper. Consider a mechanical system with configuration space Q (assumed to be an n-dimensional smooth manifold with local coordinates q) and a Lagrangian $L : TQ \to \mathbb{R}$. We will treat this system in the presence of a one-dimensional, holonomic, unilateral constraint defined by a smooth, analytic function $\phi_u : Q \to \mathbb{R}$ such that the feasible space of the system is $C = \{q \in Q \,|\, \phi_u(q) \geq 0\}$. We assume C is a submanifold with boundary in Q. Furthermore, we assume that 0 is a regular point of ϕ_u such that the boundary of C, $\partial C = \phi_u^{-1}(0)$, is a submanifold of codimension 1 in Q. Physically, ∂C is the set of contact configurations.

In accordance with the approach of [12], [13], the variational impact mechanics for the Lagrangian system above are derived as follows. A space-time formulation of Hamilton's principle

$$\delta \int_0^1 L\big(q(t), \dot{q}(t)\big) dt = 0, \tag{1}$$

is applied using a nonsmooth path space. For our purposes, we need only know that paths $q(t) \in C$ in this space are piecewise C^2 and they contain one singularity at time t_i at which $q(t_i) \in \partial C$. The stationarity conditions resulting from variations $\delta q(t)$ are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \tag{2}$$

for all $t \in [0, 1] \backslash t_i$,

$$\left[\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right]_{t_i^-}^{t_i^+} = 0, \tag{3}$$

at $t = t_i$, and

$$\left[-\frac{\partial L}{\partial \dot{q}}\right]_{t_i^-}^{t_i^+} \cdot \delta q(t_i) = 0, \tag{4}$$

for all variations $\delta q(t_i) \in T\partial C$. Qualitatively, equation (2) indicates the system obeys the standard Euler-Lagrange equations everywhere away from the impact time, t_i. At the time of impact, equations (3) and (4) imply conservation of energy and conservation of momentum tangent to the impact surface, respectively. Unsurprisingly, these are the standard conditions describing an elastic impact.

B. Nonsmooth Mechanics via Projections

In [14], an alternative variational approach is presented. This approach uses a path space of smooth curves on the whole of Q, the same space utilized in the traditional Hamilton's principle for smooth dynamics. Nonsmooth behaviors are captured, rather than in the path space, with a projection mapping $P : Q \to C$. Specifically, using a projection P in the set of mappings

$$\begin{aligned}\mathscr{P} = \{P : Q \to C \mid P(P(z)) = P(z),\, P \text{ is } C^0 \text{ on } Q,\\ P|_C(z) = z,\, P|_{Q\backslash C} \text{ is a } C^2\text{-diffeomorphism}\},\end{aligned}$$

[14] examines the projected Hamilton's principle

$$\delta \int_0^1 L\big(P(z(t)), P'(z(t))\dot{z}(t)\big) dt = 0, \qquad (5)$$

where $z(t) \in Q$ is a smooth trajectory[1] (that potentially enters the infeasible space $Q \backslash C$) and P' signifies the Jacobian of P. The stationarity conditions resulting from variations $\delta z(t)$ are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \qquad (6)$$

for all $t \in [0,1] \backslash t_i$, and

$$\left[-\frac{\partial L}{\partial \dot{q}} P' \right]_{t_i^-}^{t_i^+} = 0, \qquad (7)$$

where all instances of $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}}$ are evaluated at $(P(z(t)), P'(z(t))\dot{z}(t))$ and all instances of P' are evaluated at $z(t)$.

If we can identify conditions under which the PHP and the variational principle (1) are equivalent[2], the implications are profound. Qualitatively, the dynamics (2), (3), (4) are those of a hybrid system with resets [18], [19]. In contrast, remembering that $z(t)$ is smooth, the dynamics (6), (7) are those of a switched system [20], [17]. This means that representing nonsmooth mechanical systems with the PHP enables the use of the large body of theory and results pertaining to the control of switched systems. Thus, in the following we review sufficient conditions on P that yield the PHP (5) equivalent to the variational principle (1).

For any $P \in \mathscr{P}$, there is a trivial equivalence between $P(z(t))$ satisfying (6) and (2). For $z(t)$ satisfying (7) to yield $P(z(t))$ satisfying (3), (4), it is sufficient to require

$$P'(z(t_i^+)) \cdot \delta z_i = \delta z_i, \qquad (8)$$

for all $\delta z_i \in T_{z_i}\partial C$, and

$$\Big[-L\big(P(z), P'(z)\dot{z}\big) \Big]_{t_i^-}^{t_i^+} = 0. \qquad (9)$$

To further explore these conditions, let us assume $Q = \mathbb{R}^n$ and L is of the form

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \qquad (10)$$

where $M(q)$ is a symmetric positive definite mass matrix and $V(q)$ is a potential function. Further, assume

$$P'(z(t_i^+)) = \mathbb{I} - 2\frac{M^{-1}(\phi_u')^T \phi_u'}{\phi_u' M^{-1}(\phi_u')^T}, \qquad \forall z(t_i) \in \partial C, \qquad (11)$$

where all instances of ϕ_u' and M^{-1} are evaluated at the argument $z(t_i)$ and $\mathbb{I}$ signifies the $n \times n$ identity matrix. As discussed in [14], the assumptions (10) and (11) are sufficient to guarantee (8), (9) and consequently (3), (4) as well. The task remains, though, to identify $P \in \mathscr{P}$ satisfying (11). If such a P is defined, then the results of the principle (5) are equivalent to those of (1) for systems of the form (10). Design of a projection that globally satisfies the desired condition (11) is the subject of the following section.

III. DEFINING A GLOBALLY VALID PROJECTION

In this section, we present additional conditions by which we can define, globally on Q, a projection $P \in \mathscr{P}$ satisfying (11). Facilitating our design is the use of a specified set of normal form coordinates for the system. The resulting global projection in these coordinates comes in contrast to the projection designed in [14], which utilizes knowledge of a given impact configuration z_i in the definition of P, and is only guaranteed to act as a projection local to the specified z_i.

A. A Normal Form based on Monotonicity in ϕ_u

We begin the design of P with a transformation of the given system's coordinates. Specifically, the properties required of P are more simply viewed if we use the constraint function itself, ϕ_u, as a coordinate. A given system may not permit ϕ_u as a generalized coordinate in all cases; however, it would allow it under the following condition. Let us assume that there exists a coordinate, w.l.o.g. say q^1, such that ϕ_u is monotonic in q^1 with

$$\frac{\partial \phi_u}{\partial q^1} > 0, \qquad (12)$$

on all of Q. Then, by the implicit function theorem, the system permits as coordinates

$$\bar{q} = \begin{bmatrix} \phi_u & q^2 & \dots & q^n \end{bmatrix}^T.$$

If we denote this coordinate transformation as $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\Psi(q) = \bar{q}$, then a given system model transforms as

$$\begin{aligned} \bar{M} &= \left(\Psi'\right)^{-T} M \left(\Psi'\right)^{-1}, \\ L &= \frac{1}{2}\dot{\bar{q}}^T \bar{M} \dot{\bar{q}} - V, \\ \bar{\phi}_u &= \bar{q}^1, \\ \bar{\phi}_u' &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}. \end{aligned}$$

Qualitatively, using $\bar{q}$ coordinates linearizes ϕ_u and naturally partitions the overlying Q into respective feasible and infeasible half spaces. Topologically, this means that only systems in which the constraint surface is isomorphic to a plane will permit this coordinate transformation. For systems that meet this condition, we define a global P in normal form coordinates in the following subsection.

B. The Global P

Given that $P \in \mathscr{P}$ fully specifies $P|_C$ as the identity, to complete P we need only some $P|_{Q\backslash C} : Q \backslash C \to C \backslash \partial C$ that is a C^2-diffeomorphism and that yields (11). Note that in $\bar{q}$-coordinates (11) reduces to

$$P'(\bar{z}(t_i^+)) = \mathbb{I} - \frac{2}{(\bar{M}^{-1})_{11}} \begin{bmatrix} \left(\bar{M}^{-1}\right)_{:1} & 0_{n \times n-1} \end{bmatrix}, \qquad (13)$$

[1]For an ideal comparison with subsection II-A, here we assume $z(t) \in \partial C$ only once at time $t = t_i$. In generality, this need not be the case.

[2]That is, show $z(t)$ satisfying (6), (7) yields $P(z(t))$ a stationary path of (1).

where $\left(\bar{M}^{-1}\right)_{:1}$ denotes the first column of $\bar{M}^{-1}$. The $P|_{Q\setminus C}$ we have defined to achieve this condition is characterized by

$$\bar{z}^1 \mapsto -\bar{z}^1, \tag{14}$$

and for all $i \neq 1$,

$$\bar{z}^i \mapsto \bar{z}^i - \frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\Delta_i, \tag{15}$$

where k is a positive constant to be defined and $\Delta : Q \to \mathbb{R}^{n-1}$ is defined by

$$\Delta_i = \frac{\left(\bar{M}^{-1}\right)_{i1}}{\left(\bar{M}^{-1}\right)_{11}}. \tag{16}$$

It is straightforward to verify that (14), (15) yield the desired behavior near the boundary ∂C. One can see that as $\bar{z}^1 \to 0$, $P|_{Q\setminus C}$ approaches the identity and also $P'(\bar{z}(t_i^+))$ is that of (13). This holds regardless of the value of k. Qualitatively, the constant k serves to drive P closer to a pure reflection over the planar ∂C, while leaving $P'(\bar{z}(t_i^+))$ unchanged. We utilize this fact in the following lemma which verifies that, for appropriately large k, the mapping (14), (15) constitutes a C^2-diffeomorphism onto $C\setminus\partial C$.

Lemma 1: Given the following:

- a Lagrangian of the form (10) on $Q = \mathbb{R}^n$,
- a boundary ∂C of codimension-one with corresponding unilateral constraint ϕ_u satisfying (12),
- M is a C^2 global isomorphism on TQ,
- The linear operator[3] $\Lambda = \partial\Delta/\partial(\bar{q}^2, \ldots, \bar{q}^n)$ is continuous,

there exists $k_c \in \mathbb{R}^+$ such that for all $k > k_c$, equations (14), (15) constitute a C^2-diffeomorphism $P|_{Q\setminus C} : Q\setminus C \to C\setminus\partial C$.

Proof: We begin by computing $P'|_{Q\setminus C}$ as

$$P'|_{Q\setminus C} = \begin{bmatrix} -1 & 0 \\ \frac{k\left(\bar{z}^1\right)^2-1}{\left(1+k\left(\bar{z}^1\right)^2\right)^2}\Delta - \frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\frac{\partial\Delta}{\partial\bar{z}^1} & \mathbb{I} - \frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\Lambda \end{bmatrix},$$

which reveals

$$\det\left(P'|_{Q\setminus C}\right) = -\det\left(\mathbb{I} - \frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\Lambda\right).$$

Essentially, $P|_{Q\setminus C}$ is singular iff the right hand side above vanishes. However, by the continuity of Λ there necessarily exists $a \in \mathbb{R}^+$ such that for all $\upsilon \in \mathbb{R}^{n-1}$,

$$\|\Lambda\upsilon\| \leq a\|\upsilon\|,$$

where $\|\cdot\|$ is the standard Euclidean norm. Also, the scalar function $\frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}$ attains a maximum value of $\frac{1}{\sqrt{k}}$ at $\bar{z}^1 = \frac{1}{\sqrt{k}}$. Thus for all $\upsilon \in \mathbb{R}^{n-1}$,

$$\begin{aligned}\upsilon^T\left(\mathbb{I} - \frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\Lambda\right)\upsilon &= \|\upsilon\|^2 - \frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\upsilon^T\Lambda\upsilon, \\ &\geq \left(1 - \frac{a}{\sqrt{k}}\right)\|\upsilon\|^2.\end{aligned}$$

[3]Note, the operator Λ defined here is simply the last $n-1$ columns of the Jacobian Δ'.

Now we see that if $k > k_c = a^2$, then $\mathbb{I} - \frac{2\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\Lambda$ is positive definite and necessarily invertible. This condition yields that $P'|_{Q\setminus C}$ is never singular, and thus invertible on $Q\setminus C$. The C^2 nature of $P|_{Q\setminus C}$ comes directly from the given condition on M. In summary, with the given information (14), (15) constitute a C^2 mapping with an everywhere invertible Jacobian and both domain, $Q\setminus C$, and range, $C\setminus\partial C$, open and simply connected. Therefore, this definition of $P|_{Q\setminus C}$ is a C^2 diffeomorphism. ■

C. Example: Rigid Bar Impacting a Flat Surface

Consider the rigid bar in the plane, of length L_b and mass m_b, with one tip unilaterally constrained by a flat surface. This system is characterized by

$$\begin{aligned} q &= \begin{bmatrix} y & x & \theta \end{bmatrix}^T, \\ M &= \operatorname{diag}\left(m_b,\, m_b,\, \frac{1}{12}m_b L_b^2\right), \\ \phi_u &= y - \frac{L_b}{2}\mathrm{c}_\theta, \end{aligned}$$

where we have introduced the shorthand c_θ for $\cos\theta$ (similarly s_θ will be used for $\sin\theta$). Since ϕ_u is monotonic (actually linear) in y, we transform the system to coordinates $\bar{q} = \begin{bmatrix} \phi_u & x & \theta \end{bmatrix}^T$ where

$$\begin{aligned} \Psi' &= \begin{bmatrix} 1 & 0 & \frac{L_b}{2}\mathrm{s}_\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \bar{M} &= m_b \begin{bmatrix} 1 & 0 & -\frac{L_b}{2}\mathrm{s}_\theta \\ 0 & 1 & 0 \\ -\frac{L_b}{2}\mathrm{s}_\theta & 0 & -\frac{L_b^2}{24}\left(3\mathrm{c}_{2\theta} - 5\right) \end{bmatrix}, \\ \left(\bar{M}^{-1}\right)_{:1} &= \frac{1}{m_b}\begin{bmatrix} \frac{1}{2}\left(5 - 3\mathrm{c}_{2\theta}\right) & 0 & \frac{6\mathrm{s}_\theta}{L_b} \end{bmatrix}^T. \end{aligned}$$

Now, to identify the lower bound for k, we calculate

$$\begin{aligned} \Lambda &= \frac{\partial}{\partial(x,\theta)}\begin{bmatrix} 0 \\ \frac{12\mathrm{s}_\theta}{L_b(5-3\mathrm{c}_{2\theta})} \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \frac{6(\mathrm{c}_\theta + 3\mathrm{c}_{3\theta})}{L_b(5-3\mathrm{c}_{2\theta})^2} \end{bmatrix}. \end{aligned}$$

For this Λ we have $a = \frac{6}{L_b}$, and thus for any $k > \frac{36}{L_b^2}$ the projection

$$P(\bar{z}) = \begin{cases} \bar{z}, & \bar{z}^1 \geq 0, \\ \bar{z} - 2\begin{bmatrix} \bar{z}^1 \\ 0 \\ \frac{\bar{z}^1}{1+k\left(\bar{z}^1\right)^2}\frac{12\mathrm{s}_{\bar{z}^3}}{L_b\left(5-3\mathrm{c}_{2\bar{z}^3}\right)} \end{bmatrix}, & \bar{z}^1 < 0, \end{cases}$$

will correctly generate the bar's nonsmooth mechanics via the PHP.

IV. VARIATIONAL CONSTRAINED NONSMOOTH MECHANICS

For systems that do not fit the normal form of subsection III-A, one may make progress using constrained coordinates.

With this in mind, we will review existing results regarding variational nonsmooth Lagrangian mechanics in the presence of holonomic constraints and compare them to a constrained version of the PHP. We will discuss conditions under which one may reuse the projection of subsection III-B as part of this constrained PHP.

A. Constrained Nonsmooth Mechanics via a Nonsmooth Path Space

Consider the system model of subsection II-A in the presence of an m-dimensional holonomic constraint, where $m < n$. Assume the constraint is defined by a smooth, analytic function $\phi_h : Q \to \mathbb{R}^m$, and that $0 \in \mathbb{R}^m$ is a regular point of ϕ_h such that $N = \phi_h^{-1}(0)$ is a submanifold in Q. Further, assume that N is nowhere tangent to ∂C. In this situation, denote the feasible space (configurations obeying both unilateral and holonomic constraints) as $R = C \cap N$ and its boundary as $\partial R = \partial C \cap N$.

In accordance with the approach of [21], the variational impact mechanics of the constrained Lagrangian system above are derived using a vakonomic approach [22], a nonsmooth path space, and the space-time Hamilton's principle

$$\delta \int_0^1 \left[L\big(q(t), \dot{q}(t)\big) - (\lambda_h(t))^T \phi_h'(q(t))\dot{q}(t) \right] dt = 0, \tag{17}$$

where $\lambda_h(t)$ is an m-dimensional vector of Lagrange multipliers. The stationarity conditions resulting from (17) are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = -\dot{\lambda}_h^T \phi_h', \tag{18}$$

$$\phi_h' \dot{q} = 0, \tag{19}$$

for all $t \in [0,1] \backslash t_i$,

$$\left[\frac{\partial L}{\partial \dot{q}} \dot{q} - L \right]_{t_i^-}^{t_i^+} = 0, \tag{20}$$

$$\phi_h' \dot{q}\big|_{t_i^+} = 0, \tag{21}$$

at $t = t_i$, and

$$\left[-\frac{\partial L}{\partial \dot{q}} \right]_{t_i^-}^{t_i^+} \cdot \delta q(t_i) = 0, \tag{22}$$

for all variations $\delta q(t_i) \in T\partial R$. Qualitatively, equations (18), (19) indicate the system obeys the standard constrained Euler-Lagrange equations everywhere away from the impact time, t_i. At the time of impact, equation (20) implies conservation of energy, equation (21) implies the post impact velocity obeys the holonomic constraint, and equation (22) implies conservation of momentum that is simultaneously tangent to the impact surface and the holonomic constraint.

B. Constrained Nonsmooth Mechanics via Projections

To extend the PHP of [14] to constrained systems, we must define a new space of admissible projections. We make use of mappings that sequentially project, first to obey the unilateral constraint and then all constraints. Specifically, let us consider a projection P_c in the set of mappings

$$\begin{aligned}\mathscr{P}_c = \{ & P_c : Q \to R \mid P_c(P_c(z)) = P_c(z),\, P_c \text{ is } C^0 \text{ on } Q, \\ & P_c|_C = P_h : C \to R \text{ is } C^2,\, P_h|_{C\backslash\partial C} : C\backslash\partial C \to R\backslash\partial R, \\ & P_h|_{\partial C} : \partial C \to \partial R,\, P_c|_{Q\backslash C} = P_h \circ P_u, \\ & P_u : Q\backslash C \to C \text{ is a } C^2 \text{ diffeomorphism}\}.\end{aligned}$$

Qualitatively, in the definition of $P_c \in \mathscr{P}_c$, the mapping P_u transforms configurations to obey the unilateral constraint (the same role played by $P \in \mathscr{P}$) and P_h further maps configurations onto the feasible portion R of the holonomic constraint manifold[4] N. The use of sequential mappings is not general (there exists maps $P_c \notin \mathscr{P}_c$ that project to R), but allows us to relate results to known projection approaches in constrained mechanics.

Using $P_c \in \mathscr{P}_c$ we define the projected Hamilton's principle for constraints,

$$\delta \int_0^1 L\big(P_c(z(t)), P_c'(z(t))\dot{z}(t)\big) dt = 0, \tag{23}$$

where $z(t) \in Q$ is a smooth trajectory.[5] The stationarity conditions resulting from variations $\delta z(t)$ are

$$\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] P_h' = 0, \tag{24}$$

for all $t \in [0,1] \backslash t_i$, and

$$\left[-\frac{\partial L}{\partial \dot{q}} P_c' \right]_{t_i^-}^{t_i^+} = 0, \tag{25}$$

where all instances of $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}}$ are evaluated at $(P_c(z(t)), P_c'(z(t))\dot{z}(t))$, all instances of P_c' are evaluated at $z(t)$, and the argument of P_h' is either $z(t)$ or $P_u(z(t))$ (depending upon the feasibility of $z(t)$). Similar to the unconstrained case, let us now examine conditions under which the principles (17), (23) yield equivalent results.

For any differentiable $P_h : C \to R$, there is an equivalence between $P_c(z(t))$ satisfying (24) and (18). This can be seen by differentiating $\phi_h(P_h(z)) = 0$ to yield

$$\text{null}\left(\phi_h'\right) = \text{range}\left(P_h'\right), \tag{26}$$

for all $z \in C$. Thus, postmultiplication of (18) by P_h' yields (24). This is essentially a higher dimensional form of the null space method [23], [24] for constrained mechanical systems. Notice, the property (26) also implies that any $P_c(z(t))$ necessarily satisfies (21).

For $P_c(z(t))$ satisfying (25) to satisfy (20), (22) as well, it is sufficient to require

$$P_c'(z(t_i^+)) \cdot \delta z_i = P_c'(z(t_i^-)) \cdot \delta z_i, \tag{27}$$

[4]Though R has a reduced dimension relative to Q, throughout the following analysis we assume elements $q \in R$ are represented with the same n-dimensional coordinates as Q. Essentially, P_c maps to the embedding of R in Q, but for brevity we have left this out of our notation.

[5]As in the unconstrained case, here we assume $z(t) \in \partial C$ only once at $t = t_i$. In generality, this need not be the case.

for all $\delta z_i \in T_{z_i}\partial C$ such that $P_c'(z(t_i^-)) \cdot \delta z_i \in T\partial R$, and

$$\left[-L\left(P_c(z), P_c'(z)\dot{z}\right)\right]_{t_i^-}^{t_i^+} = 0. \tag{28}$$

In the following lemma, we parallel the unconstrained results of [14]. That is, for systems of the form (10) we provide a sufficient condition relating $P_c'(z(t_i^-))$ and $P_c'(z(t_i^+))$ such that P_c meets (27) and (28).

Lemma 2: Assume the given:

- Q, L, ϕ_u, and M as in Lemma 1,
- a holonomic constraint submanifold $N = \phi_h^{-1}(0)$ that is nowhere tangent to ∂C,
- a projection $P_c \in \mathscr{P}_c$.

If P_c' on $Q \backslash C$ is such that for all $z(t_i) \in \partial C$

$$P_c'(z(t_i^+)) = A(z(t_i))P_c'(z(t_i^-)), \tag{29}$$

where

$$A = \mathbb{I} - 2\frac{M^{-1}H\left(\phi_u'\right)^T \phi_u' H^T}{\phi_u' H^T M^{-1} H \left(\phi_u'\right)^T},$$
$$H = \mathbb{I} - \left(\phi_h'\right)^T \left(\phi_h' M^{-1} \left(\phi_h'\right)^T\right)^{-1} \phi_h' M^{-1},$$

and all instances of ϕ_u', ϕ_h', and M^{-1} are evaluated at the argument $z(t_i)$, then P_c satisfies (27), (28).

Proof: The form of (29) arises directly from the explicit solution of equations (20), (21), (22) for systems of the form (10). Specifically, the equations simplify as

$$\dot{q}(t_i^+) = \dot{q}(t_i^-) + M^{-1}\left(\phi_h'\right)^T \hat{\lambda}_h + M^{-1}\left(\phi_u'\right)^T \hat{\lambda}_u,$$
$$\phi_h' \dot{q}(t_i^+) = 0,$$
$$\left(\dot{q}(t_i^+)\right)^T M \dot{q}(t_i^+) = \left(\dot{q}(t_i^-)\right)^T M \dot{q}(t_i^-),$$

where $\hat{\lambda}_h \in \mathbb{R}^m$ and $\hat{\lambda}_u \in \mathbb{R}$ represent magnitudes of the impulses delivered to the system by the respective constraints. These equations permit the explicit solutions $\hat{\lambda}_u = 0$, $\hat{\lambda}_h = 0$, $\dot{q}(t_i^+) = \dot{q}(t_i^-)$ and

$$\hat{\lambda}_u = -2\frac{\left(\phi_u'\right)^T \phi_u' H^T}{\phi_u' H^T M^{-1} H \left(\phi_u'\right)^T} \dot{q}(t_i^-), \tag{30}$$

$$\hat{\lambda}_h = -\left(\phi_h' M^{-1}\left(\phi_h'\right)^T\right)^{-1} \phi_h' M^{-1}\left(\phi_u'\right)^T \hat{\lambda}_u, \tag{31}$$

$$\dot{q}(t_i^+) = \left[\mathbb{I} - 2\frac{M^{-1}H\left(\phi_u'\right)^T \phi_u' H^T}{\phi_u' H^T M^{-1} H\left(\phi_u'\right)^T}\right] \dot{q}(t_i^-). \tag{32}$$

We can disregard the zero impulse solution, as its post impact velocity would carry the system out of the feasible space. Focusing on the latter solution, we see the definition of $A(z)$ arises in equation (32). To see that P_c satisfies (27), we note that $A(z)$ acts as the identity operator on the subspace $T\partial R$. So, if $\delta z_i \in T_{z_i}\partial C$ is such that $P_c'(z(t_i^-)) \cdot \delta z_i \in T\partial R$, we have

$$\begin{aligned} P_c'(z(t_i^+)) \cdot \delta z_i &= A(z_i)P_c'(z(t_i^-)) \cdot \delta z_i \\ &= P_c'(z(t_i^-)) \cdot \delta z_i. \end{aligned}$$

To see that P_c satisfies (28), we note that for systems of the form (10) this condition reduces to a conservation of kinetic energy

$$\begin{aligned} &\left(P_c'(z(t_i^+))\dot{z}(t_i^-)\right)^T M P_c'(z(t_i^+))\dot{z}(t_i^-) \\ &\qquad = \left(P_c'(z(t_i^-))\dot{z}(t_i^-)\right)^T M P_c'(z(t_i^-))\dot{z}(t_i^-). \end{aligned}$$

P_c meets this condition by the fact $A^T M A = M$. Thus, condition (29) is sufficient to imply P_c satisfies (27) and (28). ■

In summary, we have shown that any $P_c \in \mathscr{P}_c$ provides equivalence between the stationarity conditions (24) and (18). Further, under the additional condition of Lemma 2, $P_c'(z(t_i^+)) = A(z(t_i))P_c'(z(t_i^-))$, the stationarity condition (25) yields results that satisfy (20), (22). In this case, any stationary $z(t)$ in the constrained PHP (23) yields $P_c(z(t))$ stationary in the principle (17). It still remains to show that one can design $P_c \in \mathscr{P}_c$ meeting the conditions of Lemma 2. Given the difficulty of this task, we do not yet approach it in general. Rather, in the following subsection we treat a special case.

C. Separability of Constraints at Impact

The form of (29), which couples the impulsive effects of ϕ_u and ϕ_h, indicates that in the general case it will be difficult to appropriately design P_h and P_u to produce the desired $P_c'(z(t_i^+))$. However, as demonstrated with the following lemma, for certain systems the roles of the competing constraints are separable and one may use the $P \in \mathscr{P}$ of subsection III-B as P_u.

Lemma 3: Given the following:

- Q, L, ϕ_u, M, and Λ as in Lemma 1,
- $\bar{\phi}_u' \bar{M}^{-1}\left(\bar{\phi}_h'\right)^T = 0$, for all $z \in \partial C$,
- P_h fitting the requirements of $\mathscr{P}_c$ with $P_h' A = A P_h'$ for all $z \in \partial C$,

then the $P \in \mathscr{P}$ defined by (14), (15) provides $P_u : Q\backslash C \to C\backslash \partial C$. The given P_h and this P_u constitute $P_c \in \mathscr{P}_c$ that meets the central requirement in Lemma 2, (29).

Proof: Begin by noting that the given $\bar{\phi}_u' \bar{M}^{-1}\left(\bar{\phi}_h'\right)^T = 0$ implies that $H\left(\phi_u'\right)^T = \left(\phi_u'\right)^T$ and $A(z)$ simplifies to $P'(z(t_i^+))$ from equation (11). Since $P_c|_{Q\backslash C} = P_h \circ P_u$, we have for all $z \in \partial C$,

$$\begin{aligned} P_c'(z(t_i^+)) &= P_h'\left(P_u(z(t_i))\right) P_u'(z(t_i^+)) \\ &= P_h'(z(t_i))P_u'(z(t_i^+)) \\ &= P_h'(z(t_i))A(z(t_i)). \end{aligned}$$

Since $P_c|_C = P_h$ we have $P_h'(z(t_i)) = P_c'(z(t_i^-))$, and with the given commutativity of A and P_h' the above simplifies to precisely (29). ■

Admittedly, the conditions required by this lemma are restrictive. However, they are not entirely inapplicable, as seen in the following example.

D. Example: Constrained Rigid Bar Impacting a Flat Surface

Return to the rigid bar of subsection III-C. Maintaining the unilateral constraint from that example, consider that the center of mass of the bar is holonomically constrained in its horizontal position with

$$\phi_h = x - f(y).$$

The separability conditions of Lemma 3, $\bar{\phi}_u' \bar{M}^{-1} \left(\bar{\phi}_h'\right)^T = 0$ and $P_h' A = A P_h'$, are both invariant under coordinate transformations. Thus, w.l.o.g. we choose to examine them in the original q coordinates. In these coordinates we have

$$\begin{aligned}
\phi_u' &= \begin{bmatrix} 1 & 0 & -\frac{L_b}{2} s_\theta \end{bmatrix}, \\
M^{-1} &= \operatorname{diag}\left(\frac{1}{m_b}, \frac{1}{m_b}, \frac{12}{m_b L_b^2}\right), \\
\phi_h' &= \begin{bmatrix} -f'(y) & 1 & 0 \end{bmatrix}, \\
\phi_u' M^{-1} \left(\phi_h'\right)^T &= -\frac{1}{m_b} f'(y).
\end{aligned}$$

We observe that this system will only meet the orthogonality condition, $\bar{\phi}_u' \bar{M}^{-1} \left(\bar{\phi}_h'\right)^T = 0$, if $f'(y) = 0$ for all $q \in \partial C$. This is true of any $f(y)$ that is constant on the interval $[-1,1]$. Moving forward with this assumption, let us propose the use of P_h of the form

$$P_h(q) = \begin{bmatrix} y & f(y) & \theta \end{bmatrix}.$$

The above condition, $f(y)$ is constant on the interval $[-1,1]$, yields that $P_h' = \operatorname{diag}(1, 0, 1)$ for all $q \in \partial C$ and thus certainly commutes with A. This means that by Lemma 3 we have that $P_u = P$ from (14), (15) and P_h as defined above compose $P_c \in \mathscr{P}_c$ that correctly generates the nonsmooth mechanics for this system via the constrained PHP.

V. THE PHP ON COMPACT MANIFOLDS

Our current treatment of each the PHP (5) and constrained PHP (23) inherently presumes the feasible space, respectively C or R, is not compact. This can be seen as a result of our assumption that the domain of both projections, P and P_c, is $Q = \mathbb{R}^n$ and thus is not compact. One cannot expect to succeed in designing differentiable projections from a non-compact domain onto a compact feasible space without encountering singularities. Thus, progression towards treating problems with a compact feasible space will require phrasing the PHP on a compact overlying manifold Q. We leave a general treatment of this case for future work, but illustrate the general idea of the PHP on compact manifolds with the following example.

A. Example: Planar Pendulum Impacting a Flat Surface

Consider a pendulum in the plane with mass m_p and length 1. We will examine this pendulum when constrained by a vertical linear surface at $x = x_c$ where $x_c \in (-1,1)$ is constant. At first thought, it may be enticing to model this system with the constrained coordinates (x,y) and constraints

$$\begin{aligned}
\phi_u &= x - x_c, \\
\phi_h &= x^2 + y^2 - 1.
\end{aligned}$$

After all, this set of coordinates possesses the desirable monotonicity property (12) in ϕ_u. However, as evidence of the arguments regarding compactness above, there is an inherent impossibility of producing a map $P_h : C \to R$ that is C^2 on all of the half space $C = \{(x,y) \,|\, x \geq x_c\}$. To overcome this issue, we will treat this system on its true configuration manifold, $\mathbb{S}^1$.

We will do most of our work in a chart of $\mathbb{S}^1$ that maps $\theta \in (-\pi, \pi]$ to $(x,y) = (\cos\theta, \sin\theta)$. In this chart, we have

$$\phi_u = \cos\theta - x_c,$$

as well as

$$\begin{aligned}
C &= [-\arccos x_c, \arccos x_c], \\
\partial C &= \{-\arccos x_c, \arccos x_c\}.
\end{aligned}$$

Following the material in subsection II-B (prior to the assumption that $Q = \mathbb{R}^n$), we wish to identify a mapping $P \in \mathscr{P}$ such that $\theta(t)$ satisfying (7) yields $P(\theta(t))$ satisfying (3), (4). As this system has dimension $n = 1$, (4) vanishes (its dimension is $n - 1 = 0$) and the properties desired of P at impact are governed by (3) alone. With little calculation one can determine that, regardless of m_p, we require $P \in \mathscr{P}$ with $P'(\theta(t_i^+)) = -1$ for all $\theta(t_i) \in \partial C$. Qualitatively, this is because impacts applied to this scalar system simply negate the incoming velocity $\dot{\theta}(t_i^-)$.

Though it is by no means unique, one P that meets the desired conditions above is

$$P(\theta) = \begin{cases} b_1(\theta - \pi) + b_3(\theta - \pi)^3, & \theta \geq 0, \\ b_1(\theta + \pi) + b_3(\theta + \pi)^3, & \theta < 0, \end{cases}$$

where $b_1 = \frac{\pi - 4\arccos x_c}{2(\pi - \arccos x_c)}$ and $b_3 = \frac{\pi - 2\arccos x_c}{2(\pi - \arccos x_c)^3}$. This map is C^2 in the given chart and yields the expected properties

$$\begin{aligned}
P(\arccos x_c) &= \arccos x_c, \\
P'(\arccos x_c) &= -1, \\
P(-\arccos x_c) &= -\arccos x_c, \\
P'(-\arccos x_c) &= -1.
\end{aligned}$$

Further, this projection is symmetric in that $P(-\theta) = -P(\theta)$ and this implies P is also C^2 in any chart possessing an open set that contains $\theta = \pi$. Hence, P is C^2 on the entirety of $Q \backslash C$. Lastly, $P : Q \backslash C \to C \backslash \partial C$ is monotonic in θ, and this combined with its differentiability properties makes it a diffeomorphism. With all of these properties, we have verified that P is such that the PHP correctly produces the nonsmooth mechanics of the pendulum system.

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

We have defined a structured normal form for nonsmooth mechanical systems, and identified sufficient conditions under which we can guarantee systems permit this form. In normal form coordinates, we have designed a global projection mapping with which the PHP correctly generates impact dynamics globally on the configuration space Q. For systems that do not permit a transformation to normal form coordinates, we have constructed a constrained version of the PHP and identified sufficient conditions for it to correctly generate impact dynamics. Regarding systems with a compact feasible space, which our current projection mapping design cannot yet accommodate, we have discussed and presented an application of the PHP on compact configuration manifolds.

B. Future Works

In continuing to develop the generality and applicability of the PHP formulation of nonsmooth mechanics, we anticipate a variety of pursuits. For systems that meet the normal form of Section III, a sufficient foundation is in place to begin formulating optimal control methods and stochastic system models in terms of the PHP. For systems that do not permit this normal form, systems with holonomic constraints, and systems with compact feasible space we will continue to pursue minimal sufficient conditions and general projection designs by which the PHP will correctly generate impact dynamics. Lastly, we must revisit and further develop the use of the PHP as an analysis tool for discrete time simulation methods, a topic initially touched upon in [14].

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