Conditions for Uniqueness in Simultaneous Impact with Application to Mechanical Design

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***Abstract*— We present a collision resolution method based on momentum maps and show how it extends to handling multiple simultaneous collisions. Simultaneous collisions, which are common in robots that walk or climb, do not necessarily have unique outcomes, but we show that for special configurations—e.g. when the surfaces of contact are orthogonal in the appropriate sense—simultaneous impacts have unique outcomes, making them considerably easier to understand and simulate. This uniqueness helps us develop a measure of the unpredictability of the impact outcome based on the state at impact and is used for gait and mechanism design, such that a mechanism's actions are more predictable and hence controllable. As a preliminary example, we explore the configuration space at impact for a model of the RHex running robot and find optimal configurations at which the unpredictability of the impact outcome is minimized.**

I. INTRODUCTION

When it comes to impact and contact modeling, there are many methods available for solving the problem [1]–[7]. Most of these methods, in particular those making use of linear complementarity problem (LCP) solvers, are highly efficient and robust when dealing with plastic impacts even as they are simultaneous. However, relatively few methods have been proposed for dealing with elastic and non-plastic impacts [8]–[10], and even fewer were designed to work with simultaneous impacts. The issue of simultaneous impacts is, however, a rather important one. Consider the general field of walking, running and climbing robots, and in particular the RHex running robot [11]. It is very rare that impacts between the legs and the ground are such that no other leg is in contact. It is also the case that the assumption of plasticity of the impacts cannot be, in most cases, based on properties of the system or of the surfaces of contact, as large amounts of kinetic energy destroy the plasticity of most impacts—think, for example, about the different outcomes of dropping a pen on a bed and of throwing a pen at a bed.

Motivated by the previous observations and our ultimate goal of simulating and designing running robots like the RHex, we propose a model of simultaneous impact based on our previous results [12], [13] and on the iterative methods for solving simultaneous impacts already proposed in the literature [10], [14]. Our approach has the advantage of returning the correct result in cases where other methods fail, such as in the case of Newton's cradle [12]. It also possesses the ability to handle both plastic and non-plastic impacts, as we will show in section II. While we are not guaranteed unique results, we find the special cases in which unique answers are available. These cases are directly related to orthogonality of the surfaces of impact under the local Riemannian metric at the impact configuration. We often have some degree of control over this measure, through both physical mechanism design and also through gait design. We also present a measure of how the whole state of a system at impact will affect the uniqueness, provided that the configuration is such that the surfaces of impact are close to orthogonal.

Finally, we explore the configuration space at impact for a simplified model of the RHex robot. This is the same model that we use to run our dynamic simulations and for which we have verified the efficacy of the empirically devised running and walking gaits (see the attached video). We find that our non-uniqueness measure is relatively small throughout most of the configuration space, something that we attribute to good underlying mechanical design. We also find that the configurations that give the local minima in this measure correspond to impact configurations during well behaved gaits.

A. Discrete Mechanics and Variational Integrators

In this section we present a very quick overview of variational methods for describing dynamical systems. This formulation is what we use to simulate our systems before and after the collision, and thus it is important to understand how our impact model fits within the framework.

Suppose we have a system—mechanical or otherwise—described by its continuous time Lagrangian, $L(q, \dot{q}, t)$. The equations that govern this system's behavior are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0, \tag{1}$$

the well known Euler-Lagrange equations. Solving this system of differential equations for $q(t)$ and $\dot{q}(t)$ gives us the configuration of the system as a function of time. In general, the differential equations are far too complicated to solve analytically, and numerical methods have to be used, in which they are discretized and solved as a system of difference equations.

An alternative method is to apply the discretization before deriving the Euler-Lagrange equations, directly onto the Lagrangian:

$$L_d(q_i, t_i, q_{i+1}, t_{i+1}) \approx \int_{t_i}^{t_{i+1}} L(q, \dot{q}, t)\ dt. \tag{2}$$

Note that the discrete Lagrangian L_d depends only on configuration variables, and not on velocity information. There

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are several ways we can implement this approximation, and we will use the midpoint rule:

$$L_d(q_i, t_i, q_{i+1}, t_{i+1}) = \\ L\left(\frac{q_i + q_{i+1}}{2}, \frac{q_{i+1} - q_i}{t_{i+1} - t_i}, \frac{t_i + t_{i+1}}{2}\right)(t_{i+1} - t_i). \quad (3)$$

The discrete equivalent of the Euler-Lagrange equations is the set of Discrete Euler-Lagrange equations (DEL), which is a set of difference equations:

$$\mathrm{D}_3 L_d(q_{k-1}, t_{k-1}, q_k, t_k) + \mathrm{D}_1 L_d(q_k, t_k, q_{k+1}, t_{k+1}) = 0,$$

where $\mathrm{D}_i F(x_1, x_2, \dots)$ stands for "the derivative of the function F with respect to it's ith argument, x_i". Equation (I-A) can be thought of as a mapping from two known configurations q_{k-1} and q_k to an unknown one, q_{k+1}. We assume here that the time variables take on predefined values, although, as we will soon see, this needn't be the case.

A common interpretation of the DEL equations is that they enforce the conservation of *discrete momentum*, which we define through the use of the discrete momentum maps

$$\mathbb{F}^-(t_a, t_b) = \mathrm{D}_1 L_d(q_a, t_a, q_b, t_b),$$
$$\mathbb{F}^+(t_a, t_b) = \mathrm{D}_3 L_d(q_a, t_a, q_b, t_b).$$

Using this notation, the DEL equations becomes

$$\mathbb{F}^+(t_{k-1}, t_k) + \mathbb{F}^-(t_k, t_{k+1}) = 0, \quad (4)$$

which state that the discrete momentum at the end of the (t_{k-1}, t_k) interval has to equal the discrete momentum at the beginning of the following interval, (t_k, t_{k+1}).

These equations are, for all but the simplest systems, nontrivial. One could use a symbolic software package (e.g. `Mathematica`) to derive and solve them, although this might prove tedious or even impossible, depending on the size of the system. As an alternative we cite previous work in which the terms in these equations and their exact derivatives are first derived using a tree structure and then used in a Newton type root finding algorithm to solve for the unknown variables [15] We will assume that all the terms needed in any further equations and their derivatives will be calculated using this method.

II. Single Collision

The equations governing a single collision at time t_* are:

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{t_*^-}^{t_*^+} + \lambda \mathrm{D}\phi(q_*) = 0, \quad (5a)$$

$$\left[\frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L\right]_{t_*^-}^{t_*^+} = 0, \quad (5b)$$

where $q_* = q(t_*)$ is the configuration at time of impact, ϕ is the function describing the impact surface and λ is a Lagrange multiplier. Equations (5) represent the conservation of momentum tangent to the impact surface and conservation of energy. The variables are $\dot{q}(t_*^+)$ and λ, and we assume we know $\dot{q}(t_*^-)$ and q_*. From here on, we will work under the assumption that we are dealing with a simple mechanical system: potentials will not depend on the velocity, $\dot{q}$. Under this assumptions, we can write the Lagrangian as:

$$L(q, \dot{q}) = \tfrac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} - V(q),$$

where we can think of M as being a mass matrix or, alternatively,

$$M(q) = \partial_{\dot{q}\dot{q}} L(q, \dot{q}). \quad (6)$$

It is true, in general, that $M(q)$ will not depend on $\dot{q}$ (hence the notation) and that it is positive definite ($x^{\mathrm{T}} M x > 0$, $\forall x$). We also assume that coordinates were chosen to avoid any degeneracy, and as such $M(q)$ will be invertible. Under these assumptions, (5) becomes

$$\dot{q}_{*+}^{\mathrm{T}} M - \dot{q}_{*-}^{\mathrm{T}} M + \lambda \mathrm{D}\phi = 0,$$
$$\dot{q}_{*+}^{\mathrm{T}} M \dot{q}_{*+} - \dot{q}_{*-}^{\mathrm{T}} M \dot{q}_{*-} = 0,$$

where, for ease of notation, we dropped the dependency of M and ϕ of the configuration q, implicitly assuming they are evaluated at q_*, the impact configuration. We can rewrite the equations as

$$\mathbf{p}_+ = \mathbf{p}_- - \lambda \mathbf{u}, \quad (7)$$
$$\|\mathbf{p}_+\|^2 = \|\mathbf{p}_-\|^2, \quad (8)$$

where $\mathbf{p}_\pm = \dot{q}_{*\pm}^{\mathrm{T}} M$ is the momentum before and after the collision and $\mathbf{u} = \mathrm{D}\phi^{\mathrm{T}}$ is the normal to the surface of impact. Here, and for the remainder of the paper, the norms and dot products between covectors are assumed to be those defined under the local Riemannian kinetic energy metric:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} M^{-1} \mathbf{v}^{\mathrm{T}}$$
$$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

This gives

$$\lambda^2 \|\mathbf{u}\|^2 - 2\lambda \langle \mathbf{p}_-, \mathbf{u} \rangle = 0.$$

The solution $\lambda = 0$ is trivial and gives $\mathbf{p}_+ = \mathbf{p}_-$, which implies no change occurred through the collision. We know this to always be false, hence we will discard this solution. We get, finally

$$\lambda = 2\frac{\langle \mathbf{p}_-, \mathbf{u} \rangle}{\|\mathbf{u}\|^2},$$
$$\mathbf{p}_+ = \mathbf{p}_- \left(I - 2M^{-1} \frac{\mathbf{u}^{\mathrm{T}} \mathbf{u}}{\|\mathbf{u}\|^2} \right).$$

This result applies to a perfectly elastic collision. For other types of collision the energy conservation equation doesn't hold in its current form. For instance, for a perfectly plastic collision, we will want to substitute (8) by

$$\langle \mathbf{p}_+, \mathbf{u} \rangle = 0,$$

which states that the exit velocity is tangential to the surface of collision. This leads to the not very unexpected solution

$$\lambda = \frac{\langle \mathbf{p}_-, \mathbf{u} \rangle}{\|\mathbf{u}\|^2},$$
$$\mathbf{p}_+ = \mathbf{p}_- \left(I - M^{-1} \frac{\mathbf{u}^{\mathrm{T}} \mathbf{u}}{\|\mathbf{u}\|^2} \right).$$

and the energy conservation equation becomes

$$K_{+} - K_{-} + \frac{\langle \mathbf{p}_{-}, \mathbf{u} \rangle^2}{2\,\|\mathbf{u}\|^2} = 0,$$

where $K_{\pm} = \frac{1}{2}\,\|\mathbf{p}_{\pm}\|^2$. We interpret the new term as the maximum loss of energy through the interaction with the collision surface, and we name it $K_{\perp}$. This makes it easy to define a coefficient of restitution based collision, through the energy equation

$$K_{+} - K_{-} + (1-e)K_{\perp} = 0,$$

where e is the coefficient of restitution. Solving the rest of the system using this equation, we get two possible solutions

$$\lambda = \frac{(1 \pm \sqrt{e})\,\langle \mathbf{p}_{-}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2},$$
$$\mathbf{p}_{+} = \mathbf{p}_{-}\left[I - (1 \pm \sqrt{e})M^{-1}\frac{\mathbf{u}^{\mathrm{T}}\mathbf{u}}{\|\mathbf{u}\|^2}\right].$$

The $(1-\sqrt{e})$ solution can be shown to be always unfeasible. Indeed, it leads to

$$\langle \mathbf{p}_{+}, \mathbf{u} \rangle = \sqrt{e}\,\langle \mathbf{p}_{-}, \mathbf{u} \rangle < 0,$$

which is unfeasible since we assume we always start with an unfeasible entry velocity.

Considering the previous derivation, define a momentum map $\Gamma(\mathbf{u})$ thusly

$$\Gamma(\mathbf{u}) = I - (1+\sqrt{e})\frac{M^{-1}\mathbf{u}^{\mathrm{T}}\mathbf{u}}{\mathbf{u}M^{-1}\mathbf{u}^{\mathrm{T}}}$$

for some covector u. Then, we can write the generalized velocity of a system after a collision as a linear mapping of the velocity before the collision

$$\mathbf{p}_{+} = \mathbf{p}_{-}\Gamma(\mathbf{u}). \tag{9}$$

Note that in the case of discrete mechanics we will likely not have access to the velocity $\dot{q}(t)$, and we must redefine our momentum. We do this using the Legendre transformation that gives us the forward momentum in (4):

$$\mathbf{p}_{-} = \mathbb{F}^{+}(q_k, q_*).$$

Once we find the $\mathbf{p}_{+}$—e.g. using (9)—we can go back to a configuration-only representation using the backwards momentum and the same Legendre transformation:

$$\mathbb{F}^{-}(q_*, q_{k+1}) = \mathbf{p}_{+}.$$

III. Multiple Simultaneous Collisions

Suppose now that the impact occurs across two surfaces at the exact same time. Each of the surfaces can act in its normal direction with arbitrary magnitude. As such, (5) becomes

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{t_*^-}^{t_*^+} + \lambda_a \mathrm{D}\phi_a(q_*) + \lambda_b \mathrm{D}\phi_b(q_*) = 0, \tag{10a}$$

$$\left[\frac{\partial L}{\partial \dot{q}}\cdot \dot{q} - L\right]_{t_*^-}^{t_*^+} = 0, \tag{10b}$$

which has the same number of equations as (5), but one extra variable, λ_b. As a consequence, the system described by (10) will have a continuum of solutions instead of the two we found before. In most cases, one cannot obtain a unique answer by solving the problem using conservation of momentum and energy alone. However, a propagative approach, in which we solve for each surface of collision separately until a feasible result is found, gives at most a finite number of distinct results [12], [13]. Such an approach, in our case, involves applying (9) repeatedly until a feasible momentum is found

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_{-}\Gamma(\mathbf{u}_1),\\
\mathbf{p}_2 &= \mathbf{p}_1\Gamma(\mathbf{u}_2),\\
&\dots\\
\mathbf{p}_{+} &= \mathbf{p}_{n-1}\Gamma(\mathbf{u}_n),
\end{aligned}$$

where $\mathbf{u}_i$ is the normal to any surface that renders $\mathbf{p}_{i-1}$ infeasible. Note that n is *not* the actual number of surfaces, but rather the number of surfaces we must reflect over until a feasible result is found. For example, in the case of two surfaces of contact, represented by $\mathbf{u}$ and $\mathbf{v}$, we might have

$$\begin{aligned}
\mathbf{p}_{+} &= \mathbf{p}_{-}\Gamma(\mathbf{u})\Gamma(\mathbf{v}),\\
\mathbf{p}_{+} &= \mathbf{p}_{-}\Gamma(\mathbf{v})\Gamma(\mathbf{u}),\\
\mathbf{p}_{+} &= \mathbf{p}_{-}\Gamma(\mathbf{u})\Gamma(\mathbf{v})\Gamma(\mathbf{u}),\\
\mathbf{p}_{+} &= \mathbf{p}_{-}\Gamma(\mathbf{v})\Gamma(\mathbf{u})\Gamma(\mathbf{v}),\\
&\dots
\end{aligned}$$

where the number of required mappings is related to $\mathbf{p}_{-}$ and the number of collision surfaces. In general, we will have:

$$\mathbf{p}_{+} = \mathbf{p}_{-}\Gamma(\mathbf{u}_1)\Gamma(\mathbf{u}_2)\Gamma(\mathbf{u}_3)\dots. \tag{11}$$

Of course, the product above is one of matrices, and as such if we were to change the order of the terms we would not be guaranteed with the same result. In fact, for even very simple systems this will not be the case [12]. The only way to make sure that the ordering of surfaces is irrelevant is to require their corresponding matrices to commute:

$$\Gamma(\mathbf{u})\,\Gamma(\mathbf{v}) = \Gamma(\mathbf{v})\,\Gamma(\mathbf{u}) \tag{12}$$

For unit length normals, this is equivalent to

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{u}, \tag{13}$$

or

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

We will show that (13) is equivalent to the two vectors being multiples of each other. First, we can safely assume neither of the two vectors is zero. Let u_i and v_i represent the components of the vectors. Then (13) is equivalent to

$$u_i v_j = u_j v_i, \quad \forall i, j$$

which, in particular, implies

$$\frac{v_i}{u_i} = \frac{v_1}{u_1} = \alpha$$

which, in turn, implies $\mathbf{v} = \alpha\mathbf{u}$. The reverse implication is shown by calculating

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{u}^{\mathrm{T}}\alpha\mathbf{u} = \left(\alpha\mathbf{u}^{\mathrm{T}}\right)\mathbf{u} = (\alpha\mathbf{u})^{\mathrm{T}}\mathbf{u} = \mathbf{v}^{\mathrm{T}}\mathbf{u}.$$

What this means is that, for our mappings to commute we must have either

$$\mathbf{v} = \alpha\mathbf{u} \tag{14}$$

or

$$\langle \mathbf{u}, \mathbf{v}\rangle = 0. \tag{15}$$

The condition in (14) can be met only if the two surfaces are tangent at the point of contact. This is either an highly degenerate case—if the two surfaces oppose each other—or a trivial case—if the two surfaces are identical—neither of which are likely to arise in any real physical systems. The third possibility, and the one we are going to explore and exploit, is given by (15): the two surfaces are orthogonal under the local kinetic energy metric.

IV. Orthogonality

We want the mechanism to undergo multiple impacts in such a way that the result is unique (as per section III).

Let us calculate how much a deviation from orthogonality affects the result of the impact. To do this, consider two surfaces defined by the covectors $\mathbf{u}$ and $\mathbf{v}$, such that they are unit length and orthogonal, as in (15). Now let us perturb one of the vectors in a direction orthogonal to it:

$$\mathbf{v}_{\varepsilon} = \mathbf{v} + \varepsilon\mathbf{w}. \tag{16}$$

We assume, without any loss of generality that $\mathbf{w}$ is also normalized, such that $\|\mathbf{w}\| = 1$. Starting with an incoming momentum $\mathbf{p}$, the mapping to the exit momentum is

$$\mathbf{p}_{\varepsilon} = \mathbf{p}\Gamma(\mathbf{v}_{\varepsilon})\Gamma(\mathbf{u}) = \mathbf{p}\left[I - 2M^{-1}\left(\frac{\mathbf{v}_{\varepsilon}^{\mathrm{T}}\mathbf{v}_{\varepsilon} - 2\varepsilon\langle\mathbf{u},\mathbf{w}\rangle\mathbf{v}_{\varepsilon}^{\mathrm{T}}\mathbf{u}}{\|\mathbf{v}_{\varepsilon}\|^{2}} - \mathbf{u}^{\mathrm{T}}\mathbf{u}\right)\right]. \tag{17}$$

Using the ε trick to calculate the derivative, we get

$$d\mathbf{p} = \left.\frac{d\mathbf{p}_{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = 2\mathbf{p}M^{-1}\left(\mathbf{v}^{\mathrm{T}}\mathbf{w} + \mathbf{w}^{\mathrm{T}}\mathbf{v} - 2\langle\mathbf{u},\mathbf{w}\rangle\mathbf{v}^{\mathrm{T}}\mathbf{u}\right). \tag{18}$$

This is a vector that describes the change in the solution due to a small change in angle between the two surfaces of contact. As we discussed before, the reflection transformations are not commutative, which means that the order in which we take them matters. The difference would show up in this result, specifically by changing the sign of the derivative. It is not completely unexpected that this change is orthogonal to the original, unique solution:

$$\langle d\mathbf{p}, \mathbf{p}_{+}\rangle \;=\; 2\mathbf{p}M^{-1}\left(\mathbf{v}^{\mathrm{T}}\mathbf{w} - \mathbf{w}^{\mathrm{T}}\mathbf{v}\right)M^{-1}\mathbf{p}^{\mathrm{T}} \;=\; 0.$$

The reason why this is expected is that the reflection operations are distance preserving, so that any infinitesimal change in the result must occur on a direction perpendicular to the original vector. Also, let's look at the norm of dp:

$$\begin{aligned}\langle d\mathbf{p}, d\mathbf{p}\rangle &= 4\mathbf{p}M^{-1}(\mathbf{v}^{\mathrm{T}}\mathbf{v} + \mathbf{w}^{\mathrm{T}}\mathbf{w})M^{-1}\mathbf{p}^{\mathrm{T}} \\ &= 4\left[\langle\mathbf{p},\mathbf{v}\rangle^{2} + \langle\mathbf{p},\mathbf{w}\rangle^{2}\right] = 4\langle\mathbf{p}^{*},\mathbf{p}^{*}\rangle = 4\|\mathbf{p}^{*}\|^{2},\end{aligned} \tag{19}$$

where

$$\mathbf{p}^{*} = \langle\mathbf{p},\mathbf{w}\rangle\mathbf{w} + \langle\mathbf{p},\mathbf{v}\rangle\mathbf{v}. \tag{20}$$

In other words, $\mathbf{p}^{*}$ is the projection of $\mathbf{p}$ onto the subspace defined by $\mathbf{v}$ and $\mathbf{w}$, $\mathrm{Span}\{\mathbf{v},\mathbf{w}\}$.

Equation (18) gives us a design criterion for the momentum at the time of impact. Assuming we know that our surfaces are close to orthogonal we can calculate the vector $\varepsilon\mathbf{w}$ which would bring them to perfect orthogonality. Then the design decision should try and minimize $\mathbf{p}^{*}$. We expect this to be much easier to do in real systems than finding a usable configuration for which the surfaces are perfectly orthogonal (as in our example in sec.V).

V. Numerical Results for a Two Legged RHex Model

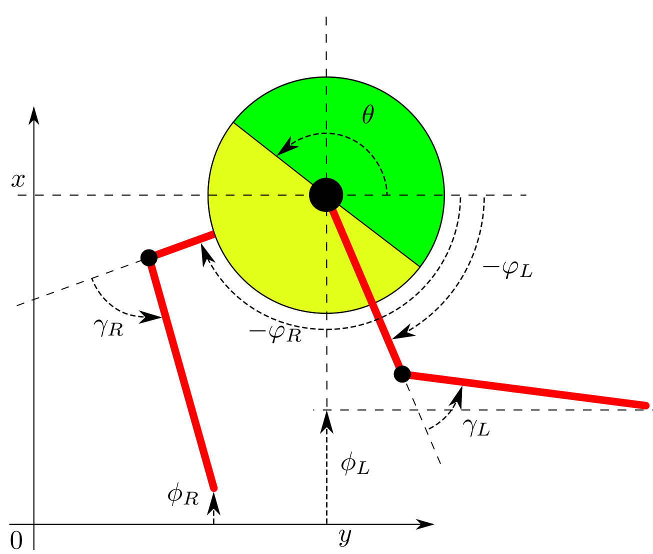

Fig. 1. The diagram of the model we used indicating all the configuration variables used to describe the system: x, y and θ are the coordinates and angle of the robot body; φ are the angles of the left and right hip; and γ are the angles at the knees. The model is actuated only at the hips and the knees are modeled with a linear spring and damper. The gap functions ϕ are the distances between the left and right foot and the ground, and their derivatives are used to define the contact surface normals in (23)

We apply the concepts presented in the previous sections to a simplified model of the RHex robot. We are interested in designing and testing different gaits in simulation. For this, our model will be a two dimensional, two legged version of the hexapod-inspired robot. A diagram with the relevant parameters is shown in fig. 1. Notice that this is a seven dimensional system, and its configuration belong to a seven dimensional manifold. We are only interested in the configurations for which both feet touch the ground, which are enforced by the two scalar constraints:

$$\phi_R(q) = \phi_L(q) = 0. \tag{21}$$

What interests us is finding a gait such that the result of a multiple impact is unique, or as close as possible to unique. This would ensure that the outcome of the impact, and hence the trajectory of the system can be better predicted and controlled.

The two constraints in (21) reduce the dimensionality of the model to five. Two of the configuration variables can be safely ignored as well, specifically the x and θ coordinates of the main body. This is because these are symmetries of the system and as such they naturally disappear from following equations. For ease in calculations we simply set them to zero. One of the constraints in (21) gives us the y coordinate as a function of the other ones. This leaves us with four variables—the knee and hip angles—and one constraint—that the feet have the same y coordinate. To this we add an artificial constraint that the knee angles have to be equal. This is not entirely unfounded, since the resting knee angles are identical, and any deviation from that angle due to bending will be small. Finally, we can represent all simultaneous contact configurations using only two variables: the knee angle and one of the hip angles. Without any loss of generality we chose the right hip angle as the free variable. The reason why we don't choose the two hip angles as the two free variables is that, given any two hip angles, there might not be one knee angle such that the feet are at the same height. However, given a hip angle and the knee angles, there is always at least one angle for the other hip such that both feet touch the floor.

We know from sec. IV that if the two surfaces of contact are orthogonal under the inverse Riemannian metric then the outcome of the impact is unique. We also have a measure of the unpredictability of the impact outcome for surfaces that are close to orthogonal (i.e. the dot product between their normals is close to zero). This measure, described in (20), depends on the direction of the incoming momentum at the time of collision relative to a perturbation from an orthogonal basis. Thus, it makes sense to look at our space of contact configurations in terms of the dot product between the two surface normals

$$s = \langle \mathbf{u}, \mathbf{v} \rangle \,, \tag{22}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the normals of the contact surfaces:

$$\mathbf{u} = \frac{\mathrm{D}\phi_R(q)}{\|\mathrm{D}\phi_R(q)\|}, \quad \mathbf{v} = \frac{\mathrm{D}\phi_L(q)}{\|\mathrm{D}\phi_L(q)\|}. \tag{23}$$

Figure 2 shows the contour plot of s as a function of the rigt hip angle and knee angle. Notice the two local minima and the local maximum at A, B, and C respectively. The configurations corresponding to the points marked on the plot are shown in fig. 3. The configurations at points A and B are, in fact, the same configuration with the

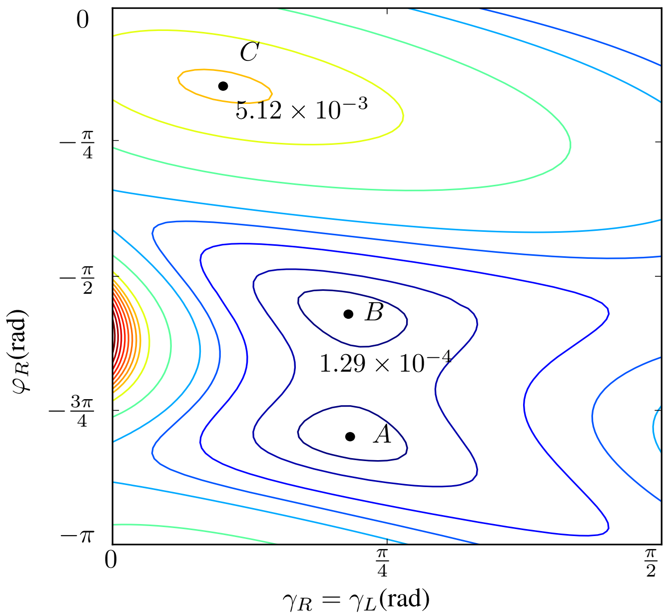

Fig. 2. The contour plot of the dot product between surface normals as a function of the configuration parameters, after the dimensionality reducing assumptions made in sec. V. Points A and B mark two local minimums (also the global ones), and point C marks a local maximum. The gaits corresponding to these points are pictured in fig. 3.

left and right legs swapped. They look very much like the impact configurations arising from a walking gait. The configuration at C is shown for comparison, and represents a local maximum. It is only a theoretical configuration, as the body and knees would penetrate the floor. However, it represents a symmetrically rotating gait, which was not found to work, either in simulation or in experiment [11]. This result gives us confidence that our non-uniqueness measure is meaningful and that it will prove useful when extended to gait and mechanism design.

VI. Conclusions and Future Work

Motivated by the goal of simulating and designing gaits for the RHex robot, we focused our attention on simultaneous impacts, which are ubiquitous in the world of walking and running robots. A fast and robust method of dealing with such situations is needed. We presented our own version of an iterative method, in which we sequentially solve for each surface that renders the momentum infeasible, until none such surface exists. We have shown that this approach will not, in general, produce unique results, since the answer is dependent on the order in which the momentum maps are considered. However, we also show that when the surfaces of contact are orthogonal in the appropriate sense, the ordering of momentum maps is irrelevant and hence the result becomes unique. We also develop a measure of how deviating from orthogonality affects the uniqueness of the result, a measure which turns out to depend on the full state of the system at impact—both configuration *and* momentum. Finally, we show numeric results for a model system based on the RHex running robot.

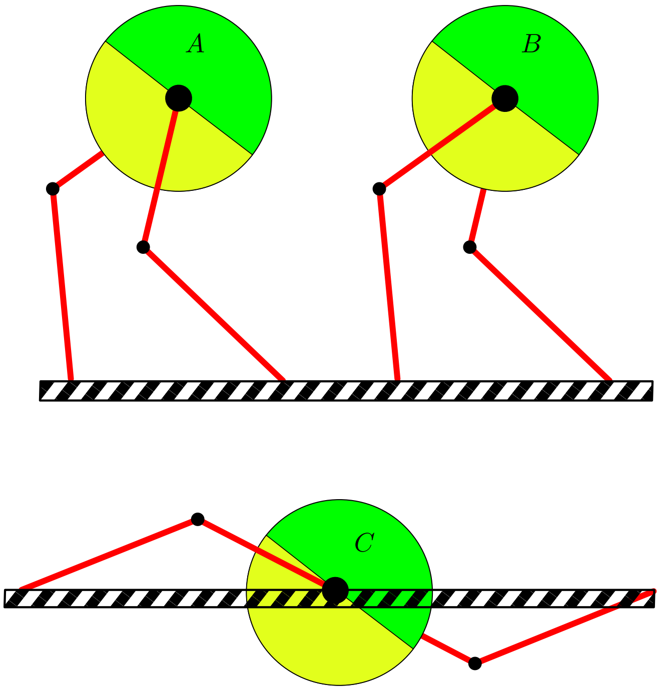

Fig. 3. The configurations corresponding to the points marked on the contour plot in fig. 2. Notice how A and B are the mirror images of the same walking stance, while C is an unrealistic stance.

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