A Projected Lagrange-d'Alembert Principle for Forced Nonsmooth Mechanics and Optimal Control

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***Abstract*— This paper extends the projected Hamilton's principle (PHP) formulation of nonsmooth mechanics to include systems with nonconservative forcing according to a projected Lagrange-d'Alembert principle (PLdAP). As seen with the conservative PHP, the PLdAP treats mechanical systems on the whole of their configuration space, captures nonsmooth behaviors using a projection mapping onto the system's feasible space, and offers additional smoothness (relative to classical approaches) in the space of solution curves. Examining implications of the PLdAP for fully actuated optimal control problems, we prove that to identify optimal feasible trajectories it is sufficient to find unconstrained trajectories according to an alternate set of optimality conditions. Focusing on the control problem expressed in the unconstrained space, we approximate optimal solutions with a path planning method that dynamically adds and removes impacts during optimization. The method is demonstrated in determining an optimal policy for a forced particle subject to a nonlinear unilateral constraint.**

I. INTRODUCTION

Control generation for nonsmooth mechanical systems is typically a difficult task, with challenges arising from systems' sensitivity to impact events. Impacts result from the presence and enforcement of unilateral constraints, and admit a variety of modeling approaches [1]. Two dominant modeling techniques are the use of measure differential inclusions (MDI) that capture impacts with complementarity conditions, and the use of hybrid system (HS) models that capture impacts in terms of guard conditions and reset maps. Use of each MDI and HS models has yielded a wealth of controls results in terms of both stability [2], [3] and optimality [4], [5]. One property these models lack, however, is their automated generation according to physical principles.

In contrast to MDI and HS are variational methods for impact mechanics [6], which generate impact models according to principles of least action. In previous works [7], [8], we have advanced a specific variational model that uses a differentiable, nonsmooth projection mapping to describe impact behaviors. As discussed in [7], [8], the dynamical system that governs "unprojected" trajectories is a switched system without resets [9] exhibiting solutions that are at minimum C^1. Thus, the projected variational representation enables one to use methods typically reserved for switched systems, or even smooth systems, to address nonsmooth problems. Upon generating solutions for the unprojected system, recovering feasible, nonsmooth solutions requires only an application of the projection mapping.

In this work, we extend the projected variational approach to incorporate nonconservative forcing according to a Projected Lagrange-d'Alembert Principle (PLdAP). This extension enables the formulation and analysis of optimal control problems under the PLdAP model. Similar to a previous result for conservative systems, we will show that first order optimality remains unaffected upon application of the projection mapping. That is, optimality of an unprojected trajectory is sufficient to imply optimality of its projected counterpart. This is a critical result, implying that we can treat optimization tasks in the unprojected space and, once solved, project them back to the feasible space while maintaining optimality. Motivated by this property, we outline the generation of optimal controls for unprojected trajectories using a finite dimensional path planning technique. There is no parallel to this technique for classical nonsmooth system models, and thus our results demonstrate the use of the PLdAP to solve nonsmooth problems with methods otherwise unavailable.

The structure of this paper is as follows. In Section II, we will review existing variational principles for forced nonsmooth mechanics [10], [11] and the conservative projected variational mechanics of [7], [8]. The section builds to the construction of the PLdAP, for which we specify conditions that yield forced solutions equivalent to the classical nonsmooth case. Section III discusses a standard optimal control problem, and the invariance of optimality through application of the projection mapping. The following Section IV outlines one particular strategy for solving the problem posed in Section III, generating optimal control designs through the use of path planning techniques. As an example of this approach, we include optimization results for a particle impacting a nonlinear collision surface. Finally, we present conclusions and future directions in Section V.

II. VARIATIONAL FORCED NONSMOOTH MECHANICS

In this section, we review the derivation of conservative and forced nonsmooth Lagrangian mechanics from variational principles. Initially, we present the classical approach seen in [6], [10], [11], which utilizes a nonsmooth path space. Next, we review the projected Hamilton's principle [7], [8] for conservative nonsmooth mechanics, and its extension to the PLdAP for the inclusion of external forces. Finally, using a comparison of respective dynamics, we provide conditions for equivalence between the varied approaches.

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A. Nonsmooth Mechanics via a Nonsmooth Path Space

To begin our discussion of nonsmooth mechanics, we establish the following system model (the same used in [7], [8]) for the remainder of the paper. Consider a mechanical system with configuration space Q (assumed to be an n-dimensional smooth manifold with local coordinates q) and a Lagrangian $L : TQ \to \mathbb{R}$. We will treat this system in the presence of a one-dimensional, holonomic, unilateral constraint defined by a smooth, analytic function $\phi_u : Q \to \mathbb{R}$ such that the feasible space of the system is $C = \{q \in Q \,|\, \phi_u(q) \geq 0\}$. We assume C is a submanifold with boundary in Q. Furthermore, we assume that 0 is a regular point of ϕ_u such that the boundary of C, $\partial C = \phi_u^{-1}(0)$, is a submanifold of codimension 1 in Q. Physically, ∂C is the set of contact configurations. When present, nonconservative external forces are represented with an exterior force field $F : \mathbb{R} \to T^*Q$.

In the conservative case, the approach seen in [6], [10] specifies that variational impact mechanics follow from utilizing a space-time formulation of Hamilton's principle. In addition to including reparameterizations of time in the space of admissible variations, the principle also makes use of nonsmooth paths in the admissible space C. Let us limit our discussion to nonsmooth paths containing a single collision, though this generalizes easily to the case of multiple isolated collisions. For our purposes, we need only know that paths $q(t) \in C$ in this space are piecewise C^2 and they contain one singularity at time t_i at which $q(t_i) \in \partial C$. With these details regarding the path space in mind, the actual statement of Hamilton's principle can remain as commonly seen

$$\delta \int_0^{T_F} L\big(q(t), \dot{q}(t)\big) dt = 0, \tag{1}$$

for some final time $T_F \in \mathbb{R}^+$. This principle yields the following relevant[1] stationarity conditions when considering admissible variations in the path space. For all $t \in [0, T_F] \backslash t_i$, the system obeys

$$D_{\text{EL}} L(q, \dot{q}, \ddot{q}) = 0, \tag{2}$$

where $D_{\text{EL}} L : \ddot{Q} \subset T(TQ) \to T^*Q$ is the Euler-Lagrange derivative. Here $\ddot{Q}$ denotes the set of second derivatives d^2q/dt^2 of paths $q(t)$, and $D_{\text{EL}} L$ is expressible coordinate-wise as

$$(D_{\text{EL}} L)_i = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right).$$

Further, at $t = t_i$ the system must satisfy

$$\left[\frac{\partial L}{\partial \dot{q}} \dot{q} - L\right]_{t_i^-}^{t_i^+} = 0, \tag{3}$$

and

$$\left[-\frac{\partial L}{\partial \dot{q}}\right]_{t_i^-}^{t_i^+} \cdot \delta q(t_i) = 0, \tag{4}$$

[1] We have excluded one stationarity condition of the principle, a redundant energy evolution equation. For full details see the aforementioned references.

for all variations $\delta q(t_i) \in T\partial C$. Qualitatively, equation (2) indicates the system obeys the standard Euler-Lagrange equations everywhere away from the impact time, t_i. At the time of impact, equations (3) and (4) imply conservation of energy and conservation of momentum tangent to the impact surface, respectively. Unsurprisingly, these are the standard conditions describing an elastic impact.

As seen in [10], [11], extending the aforementioned principle to include nonconservative forces and their associated virtual work is straightforward. Maintaining use of the same path space, a space-time formulation of the Lagrange-d'Alembert principle is stated with the common expression

$$\delta \int_0^{T_F} L\big(q(t), \dot{q}(t)\big) dt + \int_0^{T_F} F(t) \cdot \delta q(t) dt = 0. \tag{5}$$

As for the relevant stationarity conditions following from this principle, the external forcing plays no role in the impact equations (3) and (4) which remain unchanged. On the continuous time intervals away from impact, (2) is replaced by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = F. \tag{6}$$

These are the standard forced Euler-Lagrange equations.

B. Nonsmooth Mechanics via Projections

In response to the complexity of the path space used in the aforementioned principles, [7], [8] pursue an alternative variational approach. The principle therein uses a path space of curves on the whole of Q, a feature held in common with the traditional Hamilton's principle for smooth dynamics. Nonsmoothness, rather than built directly into the path space, is captured with a projection mapping $P : Q \to C$. Specifically, using a projection P in the set of mappings

$$\begin{aligned}\mathscr{P} = \{P : Q \to C \mid P(P(z)) = P(z), P \text{ is } C^0 \text{ on } Q,\\ P|_C(z) = z,\ P|_{Q\backslash C} \text{ is a } C^2\text{-diffeomorphism}\},\end{aligned}$$

[7], [8] examine the projected Hamilton's principle

$$\delta \int_0^{T_F} L\big(P(z(t)), P'(z(t))\dot{z}(t)\big) dt = 0, \tag{7}$$

where $z(t) \in Q$ and P' signifies the Jacobian of P. Notice that trajectories $z(t)$ are defined on the whole of Q and potentially enter the infeasible space $Q\backslash C$, making them nonphysical. However, in the analysis to come we shall relate them with physically meaningful trajectories $q(t) \in C$ using their image through projection map P. The stationarity conditions resulting from variations $\delta z(t)$ in (7) are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \tag{8}$$

for all $t \in [0, T_F] \backslash t_i$, and

$$\left[-\frac{\partial L}{\partial \dot{q}} P'\right]_{t_i^-}^{t_i^+} = 0, \tag{9}$$

where all instances of $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}}$ are evaluated at $(P(z(t)), P'(z(t))\dot{z}(t))$ and all instances of P' are evaluated at $z(t)$.

The inclusion of nonconservative forces in the projected principle is no more complicated than the classical case. Using the path space of curves $z(t) \in Q$, the projected Lagrange-d'Alembert principle is stated

$$0 = \delta \int_0^{T_F} L\big(P(z(t)), P'(z(t))\dot{z}(t)\big)dt + \int_0^{T_F} F(t) \cdot P'(z(t))\delta z(t) dt = 0. \quad (10)$$

Regarding the relevant stationarity conditions from this principle, in the same manner as in the classical approach the external forcing plays no role in the impact. In this case this means equation (9) still holds. On the continuous time intervals away from impact, (8) is replaced by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = F, \quad (11)$$

where again all instances of $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}}$ are evaluated at $(P(z(t)), P'(z(t))\dot{z}(t))$.

C. Establishing Equivalence between Classical and Projected Principles

Having constructed the projected principles (7) and (10), we now concern ourselves with conditions by which they adopt a physical meaning. That is to say, we present conditions on P by which solutions $z(t)$ of (7) or (10) will yield feasible $q(t) = P(z(t))$ that are stationary solutions of, respectively, (1) or (5). As mentioned in [7], [8], connecting the classical and projected principles in this way transforms the system dynamics from those of a hybrid system with resets (involving jump discontinuities in velocities $\dot{q}(t)$) to those of a switched system without resets (with trajectories that are at least C^1). The higher degree of smoothness in solutions that is gained under this transformation is a property we will leverage during optimal control design.

For the conservative case, we offer the same conditions determined in [8]. Therein, the class of systems is restricted to simple mechanical systems with a Euclidean configuration space. That is, unless stated otherwise we assume $Q = \mathbb{R}^n$ and

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \quad (12)$$

where $M(q)$ is a symmetric positive definite mass matrix and $V(q)$ is a potential function. Under these assumptions we have the following sufficient conditions for equivalence in classical and projected principles.

Lemma 1: (From [8]) Given the following:

- a Lagrangian of the form (12) on $Q = \mathbb{R}^n$,
- a projection $P \in \mathscr{P}$ that provides the boundary Jacobian[2] $P'(z(t_i^+)) = R(z(t_i))$. The desired linear map[3] $R : \partial C \to GL_n(\mathbb{R})$ is defined as

$$R(z(t_i)) = \mathbb{I} - 2\frac{M^{-1}(\phi_u')^T \phi_u'}{\phi_u' M^{-1} (\phi_u')^T}, \quad \forall z(t_i) \in \partial C, \quad (13)$$

where all instances of ϕ_u' and M^{-1} are evaluated at the argument $z(t_i)$ and $\mathbb{I}$ signifies the $n \times n$ identity matrix,

solutions $z(t)$ of (8) and (9) necessarily yield that $q(t) = P(z(t))$ satisfies (2), (3), and (4).

Proof: The equivalence between the continuous dynamics, i.e. $z(t)$ satisfying (8) and $q(t) = P(z(t))$ satisfying (2), is trivial. Given the structure of L and the Jacobian in (13), one can directly compute that $z(t)$ solving (9) is sufficient to provide

$$P'(z(t_i^+)) \cdot \delta z_i = \delta z_i, \quad (14)$$

for all $\delta z_i \in T_{z_i}\partial C$, and

$$\Big[-L\big(P(z), P'(z)\dot{z}\big)\Big]_{t_i^-}^{t_i^+} = 0. \quad (15)$$

Thus $q(t) = P(z(t))$ is a solution to (3), and (4). ∎

Given that nonconservative forcing does not influence the impact equations, we immediately obtain the following corollary.

Corollary 2: Given the conditions of Lemma 1, solutions $z(t)$ of (11) and (9) necessarily yield that $q(t) = P(z(t))$ satisfies (6), (3), and (4).

Proof: The equivalence between the continuous dynamics, i.e. $z(t)$ satisfying (11) and $q(t) = P(z(t))$ satisfying (6), is trivial. As the forcing F does not enter into the impact equations, the arguments regarding (3), (4) and (9) remain the same. ∎

III. NONSMOOTH OPTIMAL CONTROL TASKS

In this section we examine the task of optimal control design for the mechanical systems of Section II. Specifically, we define optimization problems in which we seek feasible configuration/control force pairs that minimize a given cost function. For a given subset of these control design problems, those pertaining to fully actuated mechanical systems, we determine equivalence between the optimality conditions of the classical and projected systems. This ensures that optimal controllers determined for the PLdAP equations of motion (11) and (9) remain optimal after projecting solutions back to a feasible path.

A. Optimal Control Problems for Nonsmooth Mechanical Systems

As in [12], [11], we consider the optimal control of mechanical systems as defined in terms of a cost functional J that is the integral of a performance metric $K : TQ \times T^*Q \to \mathbb{R}$. That is, for a given trajectory $q(t)$ and control force $F(t)$ the cost J is calculated as

$$J(q, F) = \int_0^{T_F} K(q(t), \dot{q}(t), F(t))dt. \quad (16)$$

[2] This is the Jacobian of P with limits evaluated on the "$Q \backslash C$ side" of the boundary ∂C. Given the definition of $\mathscr{P}$, both P and its Jacobian are the identity on the "C side" of the boundary.

[3] If it's not apparent from its structure, R defines the unique solution $\dot{q}(t_i^+) = R(q(t_i))\dot{q}(t_i^-)$ to elastic impacts for the Lagrangian (12).

Consider the task of moving a nonsmooth mechanical system between prescribed initial and final conditions during which a prescribed number of collisions, N_c, will occur. Undoubtedly, the inclusion of a priori knowledge of N_c in the task statement is undesirable and restrictive. We use it in this section to avoid considerations of mode insertion and deletion [13] when determining optimality conditions. Specifying the number of discrete events is an assumption used similarly in other works concerning hybrid optimality [14], [15]. Following the analyses in this section, we will abandon use of a given N_c when solving optimization problems in the next section. In terms of the classical equations of motion in subsection II-A, minimization of this cost during the aforementioned task requires solving the following classical system optimal control problem (CSOCP):

$$\begin{aligned} \text{Minimize} \quad & J(q,F), \\ \text{s.t.} \quad & \text{Dynamics (6) a.e. in } [0,T_F] \\ & \text{Impact eqs (4), (3) at } t_i \in (0,1),\ i \in (1,\dots,N_c), \\ & (q(0),\dot{q}(0),q(T_F),\dot{q}(T_F)) = (q_I,\dot{q}_I,q_F,\dot{q}_F), \end{aligned}$$

where the phases $(q_I,\dot{q}_I)$, $(q_F,\dot{q}_F)$ signify the given initial and final conditions, respectively.

In the same manner as for the action functionals of Section II, given a projection $P \in \mathscr{P}$ we can use the optimization problem above to induce an optimization problem in terms of trajectories $z(t)$ on the full configuration space Q by substituting $q(t) = P(z(t))$ into the cost functional J. To represent this substitution, let us establish the notation

$$\begin{aligned} \tilde{K}(z,\dot{z},F) &= K(P(z),P'(z)\dot{z},F), \\ \tilde{J}(z,F) &= J(P(z),F), \\ &= \int_0^{T_F} \tilde{K}(z(t),\dot{z}(t),F(t))dt. \end{aligned}$$

Using these terms, the projected system optimal control problem (PSOCP) is stated:

$$\begin{aligned} \text{Minimize} \quad & \tilde{J}(z,F), \\ \text{s.t.} \quad & \text{Dynamics (11) a.e. in } [0,T_F], \\ & \text{Impact eqs (9) at } t_i \in (0,T_F),\ i \in (1,\dots,N_c), \\ & (z(0),\dot{z}(0),z(T_F),\dot{z}(T_F)) = (z_I,\dot{z}_I,z_F,\dot{z}_F), \end{aligned}$$

where we have used $(z_I,\dot{z}_I)$, $(z_F,\dot{z}_F)$ to signify induced boundary conditions. In terms of P and $(q_I,\dot{q}_I)$, the induced initial conditions can be represented as $(z_I,\dot{z}_I) = (P^{-1}(q_I),(P'(z_I))^{-1}\dot{q}_I)$. These expressions hold similarly for the final conditions, simply with the index F substituted for I. Given that these induced conditions involve the inverses of P and P', and these maps are known to be only local and not global isomorphisms, the values of $(z_I,\dot{z}_I,z_F,\dot{z}_F)$ are not unique. This is an issue of practicality that we revisit when solving optimization problems in the next section.

Having defined an optimization problem for the unprojected coordinates $z(t)$, we would like to ensure that its extremals remain optimal through the projection P. That is, we wish to show that minimizers (z,F) of the PSOCP yield $(P(z),F)$ as solutions of the original problem on $C \subset Q$. While future work will surely involve deriving this sufficiency in terms of general hybrid minimum principles [16], herein we demonstrate this only for a subset of the general case. That is, moving forward in comparing optimality conditions, we work under the assumption of full actuation as described in the next subsection.

B. Fully Actuated Optimal Control Problems

Though it hasn't been apparent in our notation thus far, many control problems involve additional constraints on the control force F. These constraints can be a result of saturation, resulting in limits on the produceable magnitude of F, or underactuation, resulting in limits on the distribution of T^*Q reachable by F. In the following analysis we assume that neither of these limitations is present, and the absence of the latter has a significant impact on the structure of the specified optimization problems.

For a fully actuated system, at any configuration q the control authority exists to provide any desired $F \in T^*Q$. For the control of smooth mechanical systems, this means that every C^2 path is realizable. Further, a smooth system's dynamics as defined by the Euler-Lagrange derivative represent a map to the control force F required to produce a given q. Formally, given a C^2 path $q(t)$ the control force F that produces that path is precisely

$$F(q,\dot{q},\ddot{q}) = D_{\text{EL}}L(q,\dot{q},\ddot{q}). \tag{17}$$

Notice we've written F explicitly as a function of q and its derivatives. Regarding optimal control design, this mapping allows one to embed the constraining equations of motion into the cost function and transform the problem into one strictly in terms of the path $q(t)$. By examining the respective spaces of solution curves for our classical and projected nonsmooth mechanical systems, we can perform this same transformation on their optimal control problems when systems are fully actuated.

For the classical forced nonsmooth mechanics of the Lagrangian (12), let us consider solutions with a single impact. Similar to the results of subsection (II-A), when desired one can easily generalize the following to multiple isolated impacts. For given boundary conditions and control forces $F(t)$ that are C^0 for all $t \in [0,T_F]\backslash t_i$ (control discontinuities are allowed at the impact time, but nowhere else), solutions $q(t)$ of the dynamics (6), (3), and (4) must belong to the space of curves

$$\begin{aligned} \mathscr{Q}_{\text{sol}} = \{ q(t) : [0,T_F] \to C \,|\, & (q(0),\dot{q}(0)) = (q_I,\dot{q}_I), \\ & (q(1),\dot{q}(1)) = (q_F,\dot{q}_F),\ q(t) \text{ is } C^0, \text{ piecewise } C^2, \\ & \exists \text{ one singularity in } q(t) \text{ at } t_i,\ q(t_i) \in \partial C, \\ & \dot{q}(t_i^+) = R(q(t_i))\dot{q}(t_i^-) \}. \end{aligned}$$

Given a trajectory $q \in \mathscr{Q}_{\text{sol}}$, one can reconstruct $F(t)$ for all $t \in [0,T_F]\backslash t_i$ using (17). Keeping this in mind, we can restate the fully actuated CSOCP as

$$\begin{aligned} \text{Minimize} \quad & J(q,F(q,\dot{q},\ddot{q})), \\ \text{s.t.} \quad & q \in \mathscr{Q}_{\text{sol}}. \end{aligned}$$

This problem is expressed entirely in terms of the trajectory q, and all of the original CSOCP's constraints are built into the single requirement $q \in \mathscr{Q}_{\rm sol}$. We exclude the proof for brevity, but under our given assumptions on C and L one can show $\mathscr{Q}_{\rm sol}$ is a smooth manifold[4]. This allows us to derive first order optimality conditions for the CSOCP simply by examining variations $\delta q \in T_q\mathscr{Q}_{\rm sol}$ of solutions $q \in \mathscr{Q}_{\rm sol}$. This yields the following lemma, in which we make regular use of slot derivative notation (D_iK signifying the derivative of K w.r.t. its $i^{\rm th}$ argument):

Lemma 3: Given a fully actuated nonsmooth mechanical system with a Lagrangian of the form (12) on $Q = \mathbb{R}^n$, a pair (q,F) is a stationary point of J iff

- $F = F(q,\dot{q},\ddot{q})$ as in (17),
- away from impact, q satisfies

$$\begin{aligned} 0 =& D_1K + D_3K \cdot D_1F - \frac{d}{dt}\left(D_2K + D_3K \cdot D_2F\right) \\ &+ \frac{d^2}{dt^2}\left(D_3K \cdot D_3F\right), \end{aligned} \tag{18}$$

  for all $t \in [0,T_F]\backslash t_i$,
- at the impact time t_i, q satisfies

$$0 = -D_3K \cdot D_3F \cdot R(q) + D_3K \cdot D_3F, \tag{19}$$

$$\begin{aligned} 0 =& \left[\left(D_2K + D_3K \cdot D_2F - \frac{d}{dt}\left(D_3K \cdot D_3F\right)\right)\cdot \dot{q} - K\right]_{t_i^-}^{t_i^+} \\ &+ D_3K \cdot D_3F \cdot R'(q)\cdot(\dot{q}(t_i^-),\dot{q}(t_i^-)), \end{aligned} \tag{20}$$

$$\begin{aligned} 0 =& \left(\left[-D_2K - D_3K \cdot D_2F + \frac{d}{dt}\left(D_3K \cdot D_3F\right)\right]_{t_i^-}^{t_i^+} \right.\\ &\left. - D_3K \cdot D_3F \cdot R'(q)\cdot \dot{q}(t_i^-)\right)\cdot \delta q(t_i), \end{aligned} \tag{21}$$

  for all variations $\delta q(t_i) \in T\partial C$.

Proof: These stationarity conditions follow directly from analysis of $dJ \cdot \delta q = 0$ for all variations $\delta q \in T_q\mathscr{Q}_{\rm sol}$. The cost J, as a result of using F from (17), is a function of $(q,\dot{q},\ddot{q})$ and one must integrate by parts twice. At the impact time, substitutions are made in accordance with the relation $\delta q(t_i) = -\dot{q}(t_i)\delta t_i$ for all $\delta q(t_i) \notin T\partial C$, as well as $\delta\dot{q}(t_i^+) = R'(q(t_i))\cdot(\dot{q}(t_i^-),\delta q(t_i)) + R(q(t_i))\delta\dot{q}(t_i^-)$. The conditions (19), (20), and (21) are respectively associated with the variations $\delta\dot{q}(t_i^-) \in T(TQ)$, $\delta t_i \in \mathbb{R}$, and $\delta q(t_i) \in T\partial C$. ∎

Now consider the projection-based approach for the Lagrangian (12). Given a C^2 path $z(t)$, the control force $\tilde{F}$ associated with that path is

$$\tilde{F}(z,\dot{z},\ddot{z}) = D_{\rm EL}L\left(P(z), \frac{d}{dt}P(z), \frac{d^2}{dt^2}P(z)\right). \tag{22}$$

We've introduced the tilde notation to distinguish this mapping from (17). If we maintain the prior assumption on F (it is C^0 for all $t \neq t_i$) then solutions $z(t)$ of the dynamics (11) and (9) must belong to the space of curves

$$\begin{aligned} \mathscr{Z}_{\rm sol} = \{ & z(t) : [0,T_F] \to Q \,|\, (z(0),\dot{z}(0)) = (z_I,\dot{z}_I), \\ & (z(T_F),\dot{z}(T_F)) = (z_F,\dot{z}_F),\, z(t) \text{ is } C^1,\text{ piecewise } C^2, \\ & \exists \text{ one singularity in } \dot{z}(t) \text{ at } t_i,\, z(t_i) \in \partial C\}. \end{aligned}$$

Similar to the CSOCP, we can now recast the PSOCP as

$$\begin{aligned} \text{Minimize} \quad & \tilde{J}(z,\tilde{F}(z,\dot{z},\ddot{z})), \\ \text{s.t.} \quad & z \in \mathscr{Z}_{\rm sol}. \end{aligned}$$

Just as $\mathscr{Q}_{\rm sol}$, $\mathscr{Z}_{\rm sol}$ is a smooth manifold allowng us to derive optimality conditions using variations $\delta z \in T_z\mathscr{Z}_{\rm sol}$ of solutions $z \in \mathscr{Z}_{\rm sol}$. The following lemma results:

Lemma 4: Given a fully actuated nonsmooth mechanical system with a Lagrangian of the form (12) on $Q = \mathbb{R}^n$ and a projection $P \in \mathscr{P}$ as in Lemma 1, a pair (z,F) is a stationary point of $\tilde{J}$ iff

- $F = \tilde{F}(z,\dot{z},\ddot{z})$ as in (22),
- away from impact, z satisfies

$$\begin{aligned} 0 =& D_1\tilde{K} + D_3\tilde{K} \cdot D_1\tilde{F} - \frac{d}{dt}\left(D_2\tilde{K} + D_3\tilde{K} \cdot D_2\tilde{F}\right) \\ &+ \frac{d^2}{dt^2}\left(D_3\tilde{K} \cdot D_3\tilde{F}\right), \end{aligned} \tag{23}$$

  for all $t \in [0,T_F]\backslash t_i$,
- at the impact time t_i, z satisfies

$$0 = \left[-D_3\tilde{K} \cdot D_3\tilde{F}\right]_{t_i^-}^{t_i^+}, \tag{24}$$

$$0 = \left[\left(D_2\tilde{K} + D_3\tilde{K} \cdot D_2\tilde{F} - \frac{d}{dt}\left(D_3\tilde{K} \cdot D_3\tilde{F}\right)\right)\cdot \dot{z} - \tilde{K}\right]_{t_i^-}^{t_i^+}, \tag{25}$$

$$0 = \left[-D_2\tilde{K} - D_3\tilde{K} \cdot D_2\tilde{F} + \frac{d}{dt}\left(D_3\tilde{K} \cdot D_3\tilde{F}\right)\right]_{t_i^-}^{t_i^+} .\delta z(t_i), \tag{26}$$

  for all variations $\delta z(t_i) \in T\partial C$.

Proof: These stationarity conditions follow directly from analysis of $d\tilde{J} \cdot \delta z = 0$ for all variations $\delta z \in T_z\mathscr{Z}_{\rm sol}$. The cost $\tilde{J}$, as a result of using $\tilde{F}$ from (22), is a function of $(z,\dot{z},\ddot{z})$ and one must integrate by parts twice. At the impact time, a substitution is made in accordance with the relation $\delta z(t_i) = -\dot{z}(t_i)\delta t_i$ for all $\delta z(t_i) \notin T\partial C$. The conditions above have a slightly simpler form than those of Lemma 3, primarily due to the equality $\delta\dot{z}(t_i^+) = \delta\dot{z}(t_i^-)$. The conditions (24), (25), and (26) are respectively associated with the variations $\delta\dot{z}(t_i) \in T(TQ)$, $\delta t_i \in \mathbb{R}$, and $\delta z(t_i) \in T\partial C$. ∎

C. Equivalence between Classical and Projected Optimality Conditions

Having established the structure of the first order optimality conditions for the CSOCP and PSOCP respectively, we are able to provide the following theorem relating optimal solutions.

Theorem 5: Given a fully actuated nonsmooth mechanical system with a Lagrangian of the form (12) on $Q = \mathbb{R}^n$, a projection $P \in \mathscr{P}$ as in Lemma 1, and a stationary pair (z,F)

[4] $\mathscr{Q}_{\rm sol}$ is actually a submanifold of the path space used for the extended Hamilton's principle in [10].

satisfying the optimality conditions of Lemma 4, the pair $(P(z), F)$ satisfies the optimality conditions of Lemma 3.

Proof: Much of the correspondence between optimality conditions comes directly from the substitution $P(z) = q$. Making this substitution in equations (22) and (23) yields precisely equations (17) and (18). For the remaining optimality conditions associated with the impact time and configuration, we make use of substitutions of the form $P'(z(t_i^-)) = \mathbb{I}$, $P'(z(t_i^+)) = R(z(t_i))$, and $\dot{q}(t_i^-) = \dot{z}(t_i^-) = \dot{z}(t_i^+)$, as well as

$$\begin{aligned} D_2\tilde{F} &= D_2 D_{\mathrm{EL}} L \cdot P'(z) + 2 D_3 D_{\mathrm{EL}} L \cdot P''(z) \cdot \dot{z}, \\ D_3\tilde{F} &= D_3 D_{\mathrm{EL}} L \cdot P'(z), \end{aligned}$$

where the arguments of $D_{\mathrm{EL}}L$ are implied as $(P(z), \frac{d}{dt}P(z), \frac{d^2}{dt^2}P(z))$. Appropriately inserting these into (24), (25), and (26) yields the conditions (19), (20), and (21). ∎

Qualitatively, Theorem 5 provides that if one has determined a stationary trajectory of the PSOCP, one can immediately produce an associated stationary trajectory of the CSOCP by using the projection $P \in \mathscr{P}$. Notice, the Theorem is in terms of sufficient conditions, but makes no claims of necessity as P is not a global isomorphism.

IV. APPROXIMATE SOLUTIONS VIA PATH PLANNING

In this section, we discuss a practical approach for approximating solutions to the PSOCP. We argue that the additional degree of smoothness provided by the space of curves $\mathscr{Z}_{\mathrm{sol}}$, relative to $\mathscr{Q}_{\mathrm{sol}}$, facilitates a path planning approach to optimization. While this approach does not guarantee that solutions exactly satisfy the previous section's optimality conditions, it is simple to implement and can dynamically add and remove impacts during the optimization. We demonstrate the method on a forced planar particle system subject to a nonlinear unilateral constraint.

A. Path Planning on a Subset of $\mathscr{Z}$

Recall that, in the absence of any unilateral constraints, the solution space for smooth Lagrangian systems is simply the space of C^2 curves on Q subject to specified boundary conditions. Let us denote this space $\mathscr{C}^2(Q)$. Returning to the definition of $\mathscr{Z}_{\mathrm{sol}}$, notice its qualitative similarities to $\mathscr{C}^2(Q)$. It is only that $\mathscr{Z}_{\mathrm{sol}}$ permits isolated jump discontinuities in $\ddot{z}$ at impact events that differentiates the two[5], and in fact $\mathscr{C}^2(Q) \subset \mathscr{Z}_{\mathrm{sol}}$. This comes in stark contrast to the classical approach, for which $\mathscr{C}^2(Q) \not\subset \mathscr{Q}_{\mathrm{sol}}$ as well as $\mathscr{C}^2(C) \not\subset \mathscr{Q}_{\mathrm{sol}}$.

Essentially, $\mathscr{Z}_{\mathrm{sol}}$ is a far more permissive space than $\mathscr{Q}_{\mathrm{sol}}$, a property we can leverage in the search for optimal trajectories. Specifically, we propose the use of path planning with trajectory splines as seen in other works on optimal control [17], [18]. This entails constraining the search for optimal trajectories to the space of cubic splines in Q, a finite dimensional subspace of $\mathscr{Z}_{\mathrm{sol}}$. As this approach does not search the whole of $\mathscr{Z}_{\mathrm{sol}}$, solutions will likely be suboptimal relative to the conditions (23), (24), (25), and (26). However, the restriction to the space of splines transforms the PSOCP from an optimization over an infinite dimensional trajectory space to a finite dimensional problem solvable with direct methods.

This choice, to transform the PSOCP using a restricted trajectory space, clearly does not represent the state of the art in optimal control. Our method does demonstrate, however, fundamental advantages of the projected system approach. There is no parallel to our search over splines for the classical system. As mentioned, there is no useable subspace of smooth curves in $\mathscr{Q}_{\mathrm{sol}}$ that can capture impact behavior. Given this difference, the primary purpose of our finite dimensional path planning is to demonstrate the recovery of smooth system optimal control methods for nonsmooth problems.

That said, in generality we specify the finite dimensional, constrained PSOCP as follows. Consider a cubic spline $z_{\mathrm{cs}}(\alpha) : [0, T_F] \to Q$ where α is finite dimensional set of parameters necessary to fully specify z_{cs} on its domain $[0, T_F]$. We optimize cost as it depends on α according to

$$\begin{aligned} \text{Minimize} \quad & \hat{J}(\alpha), \\ \text{s.t.} \quad & z_{\mathrm{cs}}(\alpha) \in \mathscr{Z}_{\mathrm{sol}}, \end{aligned}$$

where $\hat{J}(\alpha) = \tilde{J}(z_{\mathrm{cs}}, \tilde{F}(z_{\mathrm{cs}}, \dot{z}_{\mathrm{cs}}, \ddot{z}_{\mathrm{cs}}))$. In our example to come there is no closed analytical expression for the integral that defines $\tilde{J}$. For such cases we recommend computing values of $\tilde{J}$, and thus $\hat{J}$ as well, using numerical quadrature.

B. Relaxation of Final Boundary Conditions

Returning to the issue of unprojected boundary conditions, recall that for a given $(q_I, \dot{q}_I)$ (resp. $(q_F, \dot{q}_F)$) there exists more than one associated $(z_I, \dot{z}_I)$ (resp. $(z_F, \dot{z}_F)$). Specifically, presuming $q_I \notin \partial C$ there is the trivial $(z_I, \dot{z}_I) = (q_I, \dot{q}_I)$ as well as a second option[6] $(z_I, \dot{z}_I) \in T(Q \backslash C)$. Notice that if $(z_I, \dot{z}_I)$ and $(z_F, \dot{z}_F)$ lie on the same side of ∂C only an even number of crossings, and thus impact events, is possible. Similarly, if $(z_I, \dot{z}_I)$ and $(z_F, \dot{z}_F)$ lie on opposite sides of ∂C an odd number of impact events must result. Essentially, given a pair of options for each $(z_I, \dot{z}_I)$ and $(z_F, \dot{z}_F)$, one's choices restrict parity in the number of resulting impacts.

As we seek to perform optimizations with no prior specifications regarding the number of impacts, we adopt the following conventions. We always use the trivial initial conditions $(z_I, \dot{z}_I) = (q_I, \dot{q}_I)$. Further, we do not choose specific values of the final conditions $(z_F, \dot{z}_F)$, but rather incorporate their influence by appending a final cost to $\hat{J}$. That is, we use

$$\hat{J}(\alpha) = \tilde{J}(z_{\mathrm{cs}}, \tilde{F}(z_{\mathrm{cs}}, \dot{z}_{\mathrm{cs}}, \ddot{z}_{\mathrm{cs}})) + K_F \|e_F\|^2$$

where $K_F \in \mathbb{R}^+$, $\|\cdot\|$ is the standard Euclidean norm, and $e_F \in \mathbb{R}^{2n}$ is the final error

$$e_F = \left(q_F - P(z_{\mathrm{cs}}(T_F)), \dot{q}_F - P'(z_{\mathrm{cs}}(T_F))\dot{z}_{\mathrm{cs}}(T_F)\right).$$

[5]Though we do not pursue it here, there are conditions on L and C by which one can show there will be no jumps in $\ddot{z}$. For such systems $\mathscr{Z}_{\mathrm{sol}} = \mathscr{C}^2(Q)$.

[6]This second case necessarily exists by the invertibility of $P|_{Q \backslash C}$, which is guaranteed in the definition of $P \in \mathscr{P}$.

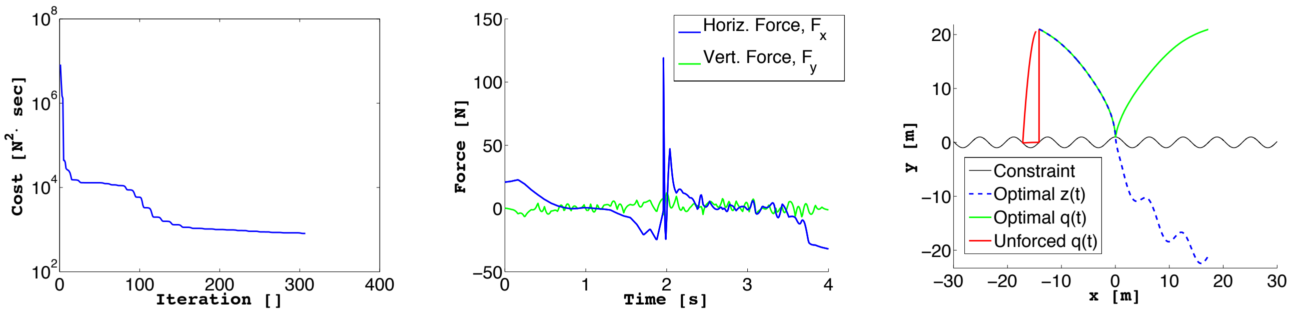

Fig. 1. Optimization results for the forced particle with nonlinear constraint surface. (Left) Using the BFGS optimization method results in monotonic cost reduction over 305 iterations. (Center) The interaction of C^2 trajectory splines and discontinuities in P'' (the Hessian of the projection) yields a jump in horizontal forces at the impact event. Vertical forces are appropriately near zero. (Right) The optimal unprojected $z(t)$ yields the optimal and feasible $q(t)$ upon application of the projection P. An unforced trajectory with equivalent initial conditions is also provided for reference.

The above convention signifies a relaxation beyond the requirements of $\mathscr{Z}_{\text{sol}}$ (as final boundary conditions are not exactly met), but with the benefit of no constraints of any kind on the number of impacts returned by our optimization.

Under relaxed final boundary conditions, we perform optimizations over a space of s-stage cubic splines in Q. We denote these splines $z_{\text{cs}}(z_0, \dot{z}_0, \ddot{z}_0, \zeta) : [0, T_F] \to Q$ where $\zeta \in \mathbb{R}^{n*s}$ contains values of constant jerk that z_{cs} exhibits on s evenly spaced intervals of $[0, T_F]$. That is,

$$\begin{aligned} z_{\text{cs}}(t) = \{ & z(t) : [0, T_F] \to Q \,|\, (z(0), \dot{z}(0), \ddot{z}(0)) = (z_0, \dot{z}_0, \ddot{z}_0), \\ & z(t) \text{ is } C^2, \text{ piecewise } C^\infty, \forall k \in \{0, \dots, s-1\}, \\ & \forall t \in [(k/s)T_F, ((k+1)/s)T_F], \dddot{z}_i(t) = \zeta_{(k*n+i)} \}. \end{aligned}$$

The option remains for users, if desired, to parameterize splines in terms of control points rather than constant jerk as we have. In the definition above we enforce $(z_0, \dot{z}_0) = (q_I, \dot{q}_I)$ so that only $\ddot{z}_0$ and ζ remain as optimization variables. That is, $\alpha = (\ddot{z}_0, \zeta)$.

C. Example: Forced Particle with Nonlinear Unilateral Constraint

To demonstrate of our algorithm for the finite dimensional PSOCP, we have generated locally optimal control inputs for a planar particle m_p impacting a nonlinear constraint surface. This system is characterized by $Q = \mathbb{R}^2$ and

$$\begin{aligned} q &= \begin{bmatrix} x & y \end{bmatrix}^T, \\ M &= m_p \mathbb{I}, \\ V(q) &= gy, \\ \phi_u &= y - \cos x. \end{aligned}$$

To express this system in terms of the PLdAP and associated dynamics, we use the global projection design of $P \in \mathscr{P}$ outlined in [8] (Sec. III B). We assume the particle is fully actuated with $F = \begin{bmatrix} F_x & F_y \end{bmatrix}^T$. We seek approximate solutions to the PSOCP when using the performance metric $K = \|F\|^2$.

Approximate solutions were determined according to a finite dimensional optimization as described in the previous subsections IV-A and IV-B. We used system parameters $m_p = 1$ [kg], $g = 10$ [kg·m/s^2], $T_F = 4$ [s], and

$$\begin{aligned} q_I &= \begin{bmatrix} -9\pi/2 & 21 \end{bmatrix}^T \text{ [m]}, \\ q_F &= \begin{bmatrix} 11\pi/2 & 21 \end{bmatrix}^T \text{ [m]}, \\ \dot{q}_I = \dot{q}_F &= \begin{bmatrix} 0 & 0 \end{bmatrix}^T \text{ [m/s]}. \end{aligned}$$

Additionally we used $s = 50$ for the number of stages in our path planning splines. Optimization was carried out using the Bruno-Fletcher-Goldfarb-Shanno (BFGS) [19] algorithm for unconstrained, nonlinear problems. Results, in of terms cost reduction, optimal input forces, and optimal trajectories are presented in Figure 1.

The observed convergence behavior (left plot), characterized by monotonic decreases of irregular magnitude, is not uncommon in nonsmooth optimization problems. For future implementations, incorporating second-order methods in place of BFGS might reduce the periods of relative small reduction seen during iterations 20-90 and 120-305. Most noticeable in the optimal control forces (center plot) is the large artificial spike in horizontal force coincident with the impact event. This results from the fact that the splines we have used are C^2 and there is a discontinuity in P'' at impact. Unless by chance the impact time was to coincide with the end of one the spline's stages, a jump in input force must result. However, the optimization does find a trajectory for which the applied vertical force is near zero. This leads to optimal trajectories (right plot) in which the feasible $q(t)$ is characterized by a horizontally forced, free fall motion, which is the expected solution for this problem.

For a closer examination of the optimization's convergence behavior, snapshots of iterative progress are provided in Figure 2. Note that when initializing the optimization our initial trajectory was simply the fixed point $q(t) = q_I$ for all $t \in [0, 4]$. From this starting guess, the Figure shows that the optimization inserts an impact event in only 2 iterations. Though we did not witness it in this particular example, we emphasize that the path planning technique used can add and remove impact events at will throughout the course of the optimization. After an impact has been added, the majority of the optimization (~ 300 iterations) is spent repositioning the impact location and smoothing the resulting $q(t)$. Notice that using the projection map designed in [8], it requires

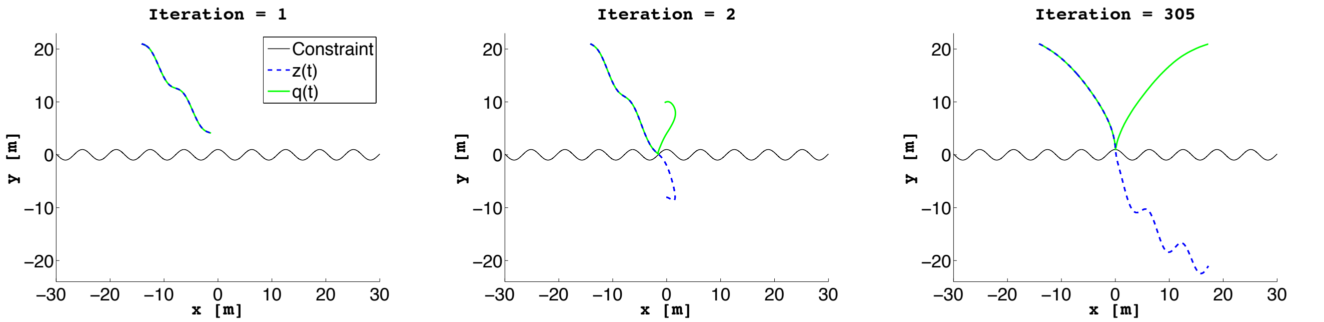

Fig. 2. Snapshots of the path planning optimization method as it converges after 305 iterations. After using a fixed point in space for an initial guess, the method rapidly inserts an impact event after only 2 iterations (left, center). Remaining iterations are spent repositioning the impact location and smoothing the overall trajectory (right).

an oscillatory motion in $z(t)$ to obtain a parabolic shape $q(t)$. Tweaking projection designs to avoid this behavior may reduce the number of iterations spent smoothing results.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

The projected variational principle representation of nonsmooth mechanical systems permits nonconservative forcing according to the PLdAP. We have identified sufficient conditions by which solutions $z(t)$ of the PLdAP are projected to yield forced, feasible, nonsmooth trajectories $q(t)$ on the constrained space C. Further we have proven that, in fully actuated cases, optimal trajectory-input pairs, $(z, \tilde{F})$, remain optimal upon projection to the feasible space. Leveraging additional smoothness of solution trajectories for the projected systems approach, we defined an optimal control generation technique based on path planning with trajectory splines. We have demonstrated the successful implementation of this method on a nonsmooth forced planar particle system, yielding approximate solutions to the PSOCP. This serves as a first example of using projected variational principles to reclaim smooth system control methods for use on nonsmooth systems.

B. Future Works

In future explorations of the PHP and PLdAP formulations of nonsmooth mechanics, we have a number of goals. We still seek new projection designs, beyond the global projection of [8], that may provide greater smoothness in solution trajectories or handle compact configuration manifolds. We must extend control analyses and algorithms to include underactuation and feedback control. Additionally, the development of the projection based approach for discrete time simulation methods and stochastic system models still remains largely open.

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