A Variational Derivation of LQR for Piecewise Time-Varying Systems

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***Abstract*— This paper provides a complete derivation for LQR optimal controllers and the optimal value function using basic principles from variational calculus. As opposed to alternatives, the derivation does not rely on the Hamilton-Jacobi-Bellman (HJB) equations, Pontryagin's Maximum Principle (PMP), or the Euler Lagrange (EL) equations. Because it requires significantly less background, the approach is educationally instructive. It provides a different perspective of how and why key quantities such as the adjoint variable and Riccati equation show up in optimal control computations and their connection to the optimal value function. Additionally, the derivation presented requires fewer regularity assumptions than necessary in applying the HJB or EL equations. As with PMP, the methods in this paper apply to systems and controls that are piecewise continuous in time.**

I. INTRODUCTION

The LQR problem concerns computation of control laws to drive linear dynamical systems along trajectories that minimize an integrated quadratic cost functional [1], [9], [15]. Solutions to this problem have been central to major developments in the field of optimal control over the last century (e.g. Kalman filtering [10], [11], Model Predictive Control [13], LQG [2], iterative nonlinear optimal control [14], etc.) and represent one of its greatest achievements. While the LQR problem can be solved using the Euler Lagrange (EL) equations and the calculus of variations [5], [6], most derivations rely on fundamental developments provided by either (or both) the Hamilton-Jacobi-Bellman (HJB) equations [3] or Pontryagin's Maximum Principle (PMP) [17], which separate the field of optimal control from the classical calculus of variations [15]. Depending on the method(s) applied, different regularity (continuity and differentiability) assumptions result.

In [1], application of the HJB equations requires a temporary regularity assumption of $\mathcal{C}^1$ (in time) system dynamics and $\mathcal{C}^2$ quadratic cost functional. After derivation, the authors indicate solutions can be tested and are provably optimal if assumptions are relaxed by one degree of continuity. In EL formulations from classical variational calculus, the problem is stated in terms of a Lagrangian function of state, velocity, and time, that must be minimized. The approach uses the fact that admissible state perturbations uniformly vanish at the optimizer (the "derivative" is 0 at the optimizer). However, classical variational calculus relies on $\mathcal{C}^1$ perturbations (the time derivative must continuously act on velocity terms) subject to endpoint constraints. The approach implies the optimal state and Lagrangian must also be $\mathcal{C}^1$. Compared to these alternatives, only PMP applies to systems with dynamics that are piecewise continuous in time and so guarantees optimality with respect to the class of switching controls.[1,2] Though relatively easy to apply, PMP requires a much more lengthy derivation than its counterparts [15].

The following section derives LQR optimal controls in a manner that allows the same degree of regularity as PMP (fewer assumptions than [1], [15] and most other popular derivations) in that it permits piecewise continuous dynamics and controls. Proofs build on basic methods from variational calculus but utilize a broader class of piecewise continuous perturbations (similar to Pontryagin's needle perturbations [15], [17]). Rather than working from the usual constrained Hamiltonian or Lagrangian formulation, we solve an unconstrained optimization using simple optimality conditions. Thus, we translate a fixed-time variable-endpoint problem solved by constrained state perturbations, to a fixed-time free-endpoint problem to be solved by unconstrained control perturbations. Posed in this different fashion, we show all the same quantities (e.g. adjoint and Riccati equations) still arise, but as the result of more direct computations. Because this approach requires less background than alternatives, it is educationally instructive and provides an alternative perspective of the necessity and role of key terms in LQR theory.

For clarity, the LQR derivation in Section II is divided into several parts. Section II-A formally states the problem including necessary assumptions. Section II-B derives an expression for the first variation of the cost and the optimal feedforward regulator[3] that sets the variation to zero for admissible perturbations. Based on continuity arguments, Section II-C proves a Riccati equation exists that translates optimal feedforward controls to feedback laws in a local neighborhood. Section II-D shows the optimal cost (value) function satisfies a simple quadratic form involving the Riccati variable in the same neighborhood, without relying on HJB formalisms. Finally, Section II-E concludes the derivation using the quadratic form of the optimal cost to show (LQR) feedback solutions are unique and exist globally. A brief discussion and conclusions are provided in Section III.

[1]In [17], Pontryagin verifies that the maximum principle can be obtained from the HJB equations but states the regularity assumptions required for derivation of the HJB equations are violated for switching solutions.

[2]The authors were surprised that they were unable to find a more direct derivation of LQ theory for this general class of systems given their importance in switched systems theory.

[3]The control will be shown to satisfy a well-known form involving the adjoint variable.

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II. Derivation of Optimal Controls

A. Problem Statement

With regard to notation, the following section refers to an arbitrary variable, Γ, with an arrow, $\vec{\Gamma}$, if it represents a vector and in bold, $\mathbf{\Gamma}$, if it refers to a matrix quantity. All other notation is standard. Applying these conventions, we address the fixed horizon optimal regulator problem from [1] for linear time-varying systems,

$$\dot{\vec{x}}(t) = \mathbf{A}(t)\,\vec{x}(t) + \mathbf{B}(t)\,\vec{u}(t)\,, \tag{1}$$

with state $\vec{x}(t) \in \mathbb{R}^{n\times 1}$ and control $\vec{u}(t) \in \mathbb{R}^{m\times 1}$. Trajectory performance is measured according to the quadratic cost functional,

$$\begin{aligned} J(t, \vec{x}(t), \vec{u}(t)) = &\int_{t_0}^{t_f} l(t, \vec{x}(t), \vec{u}(t))\,dt + m(\vec{x}(t_f)) \\ = &\ \frac{1}{2}\int_{t_0}^{t_f} \vec{x}(t)^T\mathbf{Q}(t)\,\vec{x}(t) + \vec{u}(t)^T\,\mathbf{R}(t)\,\vec{u}(t)\,dt \\ &+ \frac{1}{2}\,\vec{x}(t_f)^T\mathbf{P_1}\,\vec{x}(t_f)\,. \end{aligned} \tag{2}$$

For brevity, the time dependence of the cost functional will be dropped so $J(t, \vec{x}(t), \vec{u}(t)) \triangleq J(\vec{x}(t), \vec{u}(t))$ and $l(t, \vec{x}(t), \vec{u}(t)) \triangleq l(\vec{x}(t), \vec{u}(t))$. In deriving optimal controls, $\vec{u}^*(t)$ for $\{t \in [t_0, t_f]\,|\,t_0 < t_f\}$, that minimize (2) subject to (1), two main assumptions will be applied.

Assumption 1: $\mathbf{R}(t)$ is symmetric positive definite and $\mathbf{Q}(t)$ and $\mathbf{P_1}$ are symmetric positive semidefinite.

Assumption 2: The elements of $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are real, piecewise continuous, and bounded. Here, piecewise continuous functions refer to $\mathcal{C}^0$ functions that may contain a set of point discontinuities of Lebesgue measure zero, where the one-sided limits exist in a local neighborhood on each side. Such functions are assumed to take the value of one of their side limits at discontinuities.[4] This set of real, piecewise continuous, and bounded functions will be denoted $\widetilde{\mathcal{C}}^0$.

Assumption 1 provides a positive definite quadratic form $l(\cdot,\cdot)$ that is convex with respect to $(\vec{x}(t), \vec{u}(t))$. Because convexity is preserved under non-negative weighted integration and composition with convex equality constraints (1) (see [4]), cost (2) is also convex with respect to the control alone. This guarantees that any local extrema satisfying necessary conditions for optimality is a global extremum. For $J \in \mathcal{C}^1$, these conditions require the first cost variation vanish locally for all (admissible) perturbations to the optimal control [15]. As will be shown, Assumption 2 ensures the first variation in the cost functional is continuous (i.e. $J \in \mathcal{C}^1$) and that solutions to (1) that minimize (2) exist and are unique.

The remainder of Section II is dedicated to the proof of the well-known[5] LQR result provided in Theorem 1.

Theorem 1: Based on Assumptions 1 and 2, the control that (globally) optimizes (2) subject to (1) is provided by linear state feedback law

$$\vec{u}^*(t) = -\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\,\mathbf{P}(t)\,\vec{x}(t)\,. \tag{3}$$

This control is guaranteed to exist on $[t_0, t_f]\,\forall\,t_0 < t_f$.

B. The Optimal Feedforward Regulator

Because the state and control are equality constrained, optimal controls are often determined by minimizing (2) with respect to state and enforcing dynamic constraints (1). However, the constraints allow the state to be expressed in terms of the control as a variation of constants formula (see [8]),

$$\vec{x}(\vec{u}(t), t) = \mathbf{\Phi}(t, t_0)\vec{x}(t_0) + \int_{t_0}^{t}\mathbf{\Phi}(t, s)\mathbf{B}(s)\vec{u}(s)\,ds\,, \tag{4}$$

with a state transition matrix (STM), $\mathbf{\Phi}(\cdot,\cdot)$, which is guaranteed to exist. Following this approach, the cost functional only depends on the control, $J(\vec{u}(t))$. Therefore, optimal controls, $\vec{u}^*(t)$, result from unconstrained optimization[6]

$$\vec{u}^*(t) \triangleq \min_{\vec{u}(t)}\ J(\vec{u}(t))\,. \tag{5}$$

Because (2) is convex, it is necessary and sufficient to show the optimizer results in continuous variation,

$$\delta J = \int_{t_0}^{t_f}\frac{\delta J(\vec{u}^*(t))}{\delta\vec{u}(t)}\delta\vec{u}(t)\,dt = 0\,. \tag{6}$$

$\forall\,\delta\vec{u}(t) \in \widetilde{\mathcal{C}}^0$. Any solution satisfying this condition is globally optimal [4], [15].

Consider all perturbations $\delta\vec{u}(t) \triangleq \epsilon\,\vec{\eta}(t)$ where $\epsilon \in [0, 1)$ and $\eta(t) \in \mathbb{R}^{m\times 1}$. The optimal control satisfies

$$\begin{aligned} \delta J = &\frac{d}{d\epsilon}\int_{t_0}^{t_f} l(\vec{x}(\vec{u}^*(t) + \epsilon\,\vec{\eta}(t), t), \vec{u}^*(t) + \epsilon\,\vec{\eta}(t))\,dt\,\bigg|_{\epsilon=0} \\ &+ \frac{d}{d\epsilon}m(\vec{x}(\vec{u}^*(t) + \epsilon\,\vec{\eta}(t), t_f))\,\bigg|_{\epsilon=0} \\ = &\int_{t_0}^{t_f}\left(\frac{\partial l(\cdot,\cdot)}{\partial\vec{x}(\cdot,t)}\,\frac{d}{d\epsilon}\vec{x}(\vec{u}^*(t) + \epsilon\,\vec{\eta}(t), t)\,\bigg|_{\epsilon=0}\right. \\ &\left.+ \frac{\partial l(\cdot,\cdot)}{\partial\vec{u}(t)}\,\vec{\eta}(t)\right)dt \\ &+ \frac{\partial m(\cdot)}{\partial\vec{x}(\cdot,t_f)}\,\frac{d}{d\epsilon}\vec{x}(\vec{u}^*(t) + \epsilon\,\vec{\eta}(t), t_f)\,\bigg|_{\epsilon=0} \\ = &\int_{t_0}^{t_f}\frac{\partial l(\cdot,\cdot)}{\partial\vec{x}(\cdot,t)}\int_{t_0}^{t}\mathbf{\Phi}(t, s)\mathbf{B}(s)\vec{\eta}(s)\,ds\,dt \\ &+ \int_{t_0}^{t_f}\frac{\partial l(\cdot,\cdot)}{\partial\vec{u}(t)}\,\vec{\eta}(t)\,dt \\ &+ \frac{\partial m(\cdot)}{\partial\vec{x}(\cdot,t_f)}\int_{t_0}^{t_f}\mathbf{\Phi}(t_f, s)\mathbf{B}(s)\vec{\eta}(s)\,ds = 0\,. \end{aligned} \tag{7}$$

To simplify the double integral from the final relation in (7), partial $\frac{\partial l(\cdot,\cdot)}{\partial\vec{x}(\cdot,\cdot)}$ is incorporated into the innermost integral and the order of integration changed so that the double integral becomes

$$\int_{t_0}^{t_f}\int_{s}^{t_f}\frac{\partial l(\cdot,\cdot)}{\partial\vec{x}(\cdot,t)}\mathbf{\Phi}(t, s)\mathbf{B}(s)\vec{\eta}(s)\,dt\,ds\,. \tag{8}$$

[4]Piecewise continuous functions will be defined from the left side limit at switching points.

[5]Usually $\mathcal{C}^0$ continuity is assumed in Assumption 2.

[6]This unconstrained approach to optimization of (2) is motivated by [7].

Note that $\mathbf{B}(s)$ and perturbation direction $\vec{\eta}(s)$ do not depend on t. When these terms are pulled out of the inner integral in (8), (7) is equivalent to

$$\begin{aligned}\delta J = &\int_{t_0}^{t_f} \frac{\partial l(\cdot,\cdot)}{\partial \vec{u}(t)}\,\vec{\eta}(t)\,dt \\ &+ \int_{t_0}^{t_f}\left(\int_s^{t_f} \frac{\partial l(\cdot,\cdot)}{\partial \vec{x}(\cdot,t)}\mathbf{\Phi}(t,s)\,dt\right. \\ &\left. + \frac{\partial m(\cdot)}{\partial \vec{x}(\cdot,t_f)}\,\mathbf{\Phi}(t_f,s)\right)\mathbf{B}(s)\vec{\eta}(s)\,ds = 0\,. \qquad (9)\end{aligned}$$

Defining $\vec{\rho}(s)^T \in \mathbb{R}^{1\times n}$ as the expression inside the parenthetical above, optimal controls must satisfy

$$\delta J = \int_{t_0}^{t_f}\left(\frac{\partial l(\cdot,\cdot)}{\partial \vec{u}(s)} + \vec{\rho}(s)^T\,\mathbf{B}(s)\right)\vec{\eta}(s)\,ds = 0 \;\; \forall \vec{\eta}(s)\,. \qquad (10)$$

Asserted earlier, even for $\widetilde{\mathcal{C}}^0$ (in time) integrand, first variation (10) is absolutely continuous and yields $J \in \mathcal{C}^1$. For this case, we re-iterate that relation (10) would normally serve only as a necessary condition for a (weak) local minimizer. Convexity not only ensures local minimizers are globally optimal, but it allows consideration of more general classes of controls and perturbations (e.g. $\delta\vec{u}(t) \in \widetilde{\mathcal{C}}^0$) that satisfy first-order conditions (10).[7]

Temporarily consider the more common, restricted version of the regulator problem from [1], [15], where Assumption 2 provides $\mathcal{C}^0$ continuity and boundedness of relevant quantities.[8] In this case, admissible perturbations need only be $\mathcal{C}^0$ and unconstrained at endpoints. A standard generalization of the Fundamental Lemma of Variational Calculus (see [15], [16]), implies optimizers set $\frac{\partial l(\cdot,\cdot)}{\partial \vec{u}(s)} + \vec{\rho}(s)^T\,\mathbf{B}(s) = \vec{0}$. Based on the quadratic form (2), optimal controls would satisfy

$$\vec{u}^*(t) = -\,\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\,\vec{\rho}(t)\,. \qquad (11)$$

The feedforward control (11) matches standard results in [15] for the restricted problem assumptions.[9] However, in the full problem statement based on Assumption 2, it is not reasonable to only consider perturbations, $\vec{\eta}(t) \in \mathcal{C}^0$. This class of perturbations only guarantees optimality of controls with respect to nearby solutions obtained by such perturbations and neglects nearby $\widetilde{\mathcal{C}}^0$ control solutions (i.e. nearby switching control solutions are not considered). We will prove the above arguments in transitioning from (10) to (11) still apply if the class of admissible perturbations is generalized to $\widetilde{\mathcal{C}}^0$. To show this, an extension of the Fundamental Lemma of Variational Calculus is required.

Lemma 1: For $g(t), h(t) \in \widetilde{\mathcal{C}}^0$, if

$$\int_{t_0}^{t_f} g(t)h(t)dt = 0 \quad \forall\, h(t) \qquad (12)$$

then $g(t) = 0$ for almost all $t \in [t_0, t_f]$.

Proof: Assume $g(t) > 0$ on any finite interval $[\tau_1, \tau_2] \subseteq [t_0, t_f]$. Any choice of piecewise constant $h(t) > 0\ \forall t \in [\tau_1, \tau_2]$ and $h(t) = 0$ elsewhere violates the integral relation. A similar argument applies if $g(t) < 0$ on any finite interval. This contradiction implies $g(t) = 0$ on all finite intervals $\subseteq [t_0, t_f]$. Thus $g(t)$ may only be nonzero on a set of measure zero. This set encompasses removable discontinuities in $g(t)$. ∎

Based on the preceding lemma, (11) must be extremal at all points excluding removable discontinuities. As these controls are integrated in computing the optimal state from (1) and cost from (2), the isolated points of discontinuity (where the parenthetical in the integrand of (10) is not necessarily zero) will have no effect on state trajectory. Controls (11) are therefore optimal $\widetilde{\mathcal{C}}^0$ solutions. Note that this broader class of $\widetilde{\mathcal{C}}^0$ perturbations is ultimately unnecessary unless the quantities in Assumption 2 are in fact piecewise continuous. When this is not the case, the integral expression for $\vec{\rho}(t)$ guarantees extremal controls (11) will be $\mathcal{C}^0$. Even in this case, it is theoretically important to consider the broader class of control perturbations for the purposes of proving optimality with respect to them.

Though derived by different means (and no longer associated with a vector of Lagrange multipliers) we will now apply direct calculation to show that $\vec{\rho}(t)$ in (11) obeys the same differential equation as the adjoint variable in [15], [17]. To this end we utilize the following properties of STMs (represented by $\mathbf{\Phi}(\cdot,\cdot)$) based on $s, t, \tau \geq 0$:

1) $\frac{d}{dt}\mathbf{\Phi}(t,\tau) = \mathbf{A}(t)\mathbf{\Phi}(t,\tau)$
2) $\mathbf{\Phi}(t,s)\mathbf{\Phi}(s,\tau) = \mathbf{\Phi}(t,\tau)$
3) $\mathbf{\Phi}^{-1}(t,\tau) = \mathbf{\Phi}(\tau,t)$

These properties are proved in [8]. However, one additional required property is provided here.

Lemma 2: For a STM, $\mathbf{\Phi}(\cdot,\cdot)$, it must be true that

$$\frac{d}{d\tau}\mathbf{\Phi}(t,\tau) = -\mathbf{\Phi}(t,\tau)\mathbf{A}(\tau) \qquad (13)$$

for all $t, \tau \geq 0$.

Proof: Following from properties 2) and 3) of STMs,

$$\begin{aligned}\frac{d}{d\tau}\Big(\mathbf{\Phi}(\tau,t)\mathbf{\Phi}(t,\tau)\Big) &= \frac{d}{d\tau}\mathbf{I} \\ \mathbf{A}(\tau)\mathbf{\Phi}(\tau,t)\mathbf{\Phi}(t,\tau) + \mathbf{\Phi}(\tau,t)\frac{d}{d\tau}\mathbf{\Phi}(t,\tau) &= \mathbf{0} \\ \mathbf{A}(\tau) &= -\mathbf{\Phi}(\tau,t)\frac{d}{d\tau}\mathbf{\Phi}(t,\tau) \\ -\mathbf{\Phi}(t,\tau)\mathbf{A}(\tau) &= \frac{d}{d\tau}\mathbf{\Phi}(t,\tau)\,,\end{aligned}$$

and so (13) must be valid. ∎

Using STM property 3) and swapping integration limits, $\vec{\rho}(s)$ can be expressed as a variation of constants formula,

$$\vec{\rho}(s) = \left(\frac{\partial m(\cdot)}{\partial \vec{x}(\cdot,t_f)}\,\mathbf{\Phi}^{-1}(s,t_f) - \int_{t_f}^{s}\frac{\partial l(\cdot,\cdot)}{\partial \vec{x}(\cdot,t)}\mathbf{\Phi}^{-1}(s,t)\,dt\right)^T. \qquad (14)$$

[7]For a discussion of strong / weak extremals including how convexity can yield non-smooth global minimizers from necessary conditions see ch. 4-5 of [18].

[8]Note that, for reasons mentioned earlier, standard derivations of the EL and HJB equations only apply to this restricted problem.

[9]The variable $\vec{\rho}(t)$ will be shown to obeys the same adjoint equation that shows up in alternative (constrained optimization) formulations.

Above, $\vec{\rho}(s)$ is similar to (4) except that its STM is the transpose of $\mathbf{\Phi}^{-1}(\cdot,\cdot)$, and initial time t_0 in (4) is replaced with terminal time t_f. First, note that if STM $\mathbf{\Phi}(\cdot,\cdot)$ corresponds to a linear system with drift vector field $\mathbf{A}(t)$, the inverse STM, $\mathbf{\Phi}^{-1}(\cdot,\cdot)$, is an STM associated with opposite drift vector field $-\mathbf{A}(t)$. This inverse STM in the first term of (14) propagates a condition from terminal time t_f to the desired evaluation time, s, rather than operating on an initial condition as in (4). Similarly, the integral flows backwards from the terminal time to time s. Based on these facts, it should be apparent that (14) is the variation of constants formula for backwards ordinary differential equation

$$\dot{\vec{\rho}}(t) = -\mathbf{A}(t)^T \vec{\rho}(t) - \frac{\partial l(\cdot,\cdot)}{\partial \vec{x}(\cdot,t)}^T \tag{15}$$

with terminal condition $\vec{\rho}(t_f) = \frac{\partial m(\cdot)}{\partial \vec{x}(\cdot,t_f)}^T$. This is the adjoint equation from [15], [17].

C. Locally Optimal Feedback Solutions

Optimal controls (11) depend on $\vec{\rho}(t)$, but $\vec{\rho}(t)$ also depends on the optimal state through $\frac{\partial l(\cdot,\cdot)}{\partial \vec{x}(\cdot,t)}^T = \mathbf{Q}(t)\,\vec{x}(\vec{u}^*(t),t)$. The optimal pair, $(\vec{x},\vec{\rho})$, must therefore be computed simultaneously from the system of differential equations (1) and (15), and optimal control (11). Because this system relies on both initial and terminal conditions, it is a two-point boundary value problem (TPBVP).

Computing optimal controls (11) from the TPBVP only facilitates feedforward implementation. To develop feedback solutions that depend only on the current state (as in (3)), it is desirable to obtain a linear mapping $\vec{x}(\cdot,t) \mapsto \vec{\rho}(t)$. Such a mapping can be derived from terminal condition $\vec{\rho}(t_f) = \mathbf{P_1}\vec{x}(\cdot,t_f)$.

Lemma 3: There exists a linear relationship $\vec{\rho}(t) = \mathbf{P}(t)\vec{x}(\cdot,t)$ at least in a neighborhood of t_f.

Proof: Because the state and adjoint are obtained from integral formulas based on the $\widetilde{\mathcal{C}}^0$ matrices in Assumption 2, both the state and adjoint are absolutely continuous in time. Hence, their left side limit (and derivative) must exist for a nonzero neighborhood of t_f. In this neighborhood the linear relation between $\vec{\rho}(t)$ and $\vec{x}(\cdot,t)$ exists such that

$$\vec{\rho}(t) = \mathbf{P}(t)\,\vec{x}(\cdot,t)\,. \tag{16}$$

Differentiating both sides and applying (1) and (15) provides

$$\begin{aligned} &-\mathbf{Q}(t)\,\vec{x}(\cdot,t) - \mathbf{A}(t)^T\mathbf{P}(t)\,\vec{x}(\cdot,t) \\ &= \dot{\mathbf{P}}(t)\,\vec{x}(\cdot,t) + \mathbf{P}(t)\,(\mathbf{A}(t)\,\vec{x}(\cdot,t) + \mathbf{B}(t)\,\vec{u}^*(t))\,. \end{aligned}$$

Substituting (11) and canceling $\vec{x}(\cdot,t)$ from both sides yields the Riccati differential equation [8]

$$\begin{aligned} \mathbf{0} = \mathbf{Q}(t) + \dot{\mathbf{P}}(t) + \mathbf{A}(t)^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) \\ - \mathbf{P}(t)\,\mathbf{B}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\mathbf{P}(t)\,, \end{aligned} \tag{17}$$

with terminal condition $\mathbf{P}(t_f) = \mathbf{P_1}$. As discussed, continuity ensures solutions, $\mathbf{P}(t)$, exist and are calculable from the left of the terminal condition (for $t < t_f$), at least in an infinitesimal neighborhood of t_f. These solutions define the linear relationship in (16). ■

Because its terminal condition is symmetric and the flow of differential equation (17) is symmetric, matrix $\mathbf{P}(t)$ must also be symmetric. Additionally, $\mathbf{P}(t)$ is positive semidefinite. A direct proof that does not rely on HJB equations, dynamic programming, or PMP is provided here.[10]

Lemma 4: $\mathbf{P}(t)$ is positive semidefinite.

Proof: Assume the symmetric matrix $\mathbf{P}(t) \in \mathcal{C}^0$ is not positive semidefinite $\forall t$, but $\mathbf{P}(t_f) = \mathbf{P_1}$ is positive semidefinite. Then, $\exists\, \vec{\nu} \in \mathbb{R}^n$ and time $\tau \le t_f$ such that $\vec{\nu}^T\mathbf{P}(\tau)\,\vec{\nu} = 0$ and an $\epsilon > 0$ such that $\vec{\nu}^T\mathbf{P}(\tau-\epsilon)\,\vec{\nu} < 0$.[11] However, based on (17),

$$\begin{aligned} \vec{\nu}^T\dot{\mathbf{P}}(\tau)\,\vec{\nu} &= -\vec{\nu}^T\mathbf{A}(\tau)^T\mathbf{P}(\tau)\,\vec{\nu} - \vec{\nu}^T\mathbf{P}(\tau)\mathbf{A}(\tau)\,\vec{\nu} \\ &\quad + \vec{\nu}^T\mathbf{P}(\tau)\,\mathbf{B}(\tau)\,\mathbf{R}^{-1}(\tau)\,\mathbf{B}(\tau)^T\mathbf{P}(\tau)\,\vec{\nu} \\ &\quad - \vec{\nu}^T\mathbf{Q}(\tau)\,\vec{\nu} \\ &= -\vec{\nu}^T\mathbf{Q}(\tau)\,\vec{\nu}\ , \end{aligned} \tag{18}$$

where the time derivative can be taken from either side at a piecewise discontinuity. Because $\mathbf{Q}(\tau)$ is positive semidefinite, (18) implies that it is impossible for $\mathbf{P}(t)$ to become positive definite forward in time from time $t = \tau$. This is a contradiction so $\mathbf{P}(t)$ must be positive semidefinite $\forall t$. ■

D. The Optimal Value Function

Solutions to (17) provide the linear map required to show optimal controls (11) take on the linear feedback form in (3). To prove that solutions to (17) must exist over the entire horizon $[t_0,t_f]$, it is necessary to guarantee (17) does not exhibit finite escape. This is commonly demonstrated using an expression of the optimal cost functional in terms of (3) (see [1], [15]). We obtain this expression next by direct calculation. In the process, we show the optimal cost depends only on the initial condition, $\vec{x}(t_0)$, and initial time, t_0. For this reason, the optimal cost will be preemptively redefined as $V(t_0,x_0) \triangleq J(\vec{x}(\vec{u}^*(t),t),\vec{u}^*(t))$ and referred to as the optimal *value function* (sometimes called the optimal *cost-to-go* [15]). The name highlights that, expressed as $V(t,\vec{x}(t))$, it returns the value of the optimal cost to traverse the remaining interval $[t,t_f]$.

Lemma 5: Based on Assumptions 1 and 2, the optimal value function satisfies

$$V(t_0,\vec{x}_0) = \frac{1}{2}\vec{x}(t_0)^T\mathbf{P}(t_0)\,\vec{x}(t_0). \tag{19}$$

Proof: Similarly to how $\mathbf{\Phi}(\cdot,\cdot)$ represents the STM for open-loop system (1), there exists a STM for the closed-loop linear system,

$$\begin{aligned} \dot{\vec{x}}(\vec{u}^*(t),t) = \mathbf{A}(t)\,\vec{x}(\vec{u}^*(t),t) \\ -\mathbf{B}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\,\mathbf{P}(t)\,\vec{x}(\vec{u}^*(t),t)\,. \end{aligned} \tag{20}$$

Where $\mathbf{\Phi}(\cdot,\cdot)$ is derived from $\mathbf{A}(t)$, this closed-loop STM, which will be referred to as $\mathbf{\Psi}(\cdot,\cdot)$, is derived from drift

[10] The proof is based on arguments provided in [19]. In reviewing existing literature, the authors found nearly all well-known proofs rely on an expression for the optimal cost in terms of $\mathbf{P}(t)$.

[11] The final relation in (18) results because symmetry permits Cholesky factorization $\mathbf{P}(\tau) = \mathbf{L}^T\mathbf{L}$. Hence, $\vec{\nu}^T\mathbf{P}(\tau)\,\vec{\nu} = 0$ implies $\vec{\nu} \in \mathcal{N}(\mathbf{P}(\tau))$.

vector field $\mathbf{\Lambda}(t) \triangleq \mathbf{A}(t) - \mathbf{B}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\,\mathbf{P}(t)$. The solution to (20) is therefore

$$\vec{x}(\vec{u}^*(t), t) = \mathbf{\Psi}(t, t_0)\,\vec{x}(t_0)\,, \tag{21}$$

and the optimal control can be restated as

$$\vec{u}^*(t) = -\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\mathbf{P}(t)\,\mathbf{\Psi}(t, t_0)\,\vec{x}(t_0)\,. \tag{22}$$

From (21) and (22), the optimal value function satisfies

$$\begin{aligned} V(t_0, \vec{x}_0) = \frac{1}{2}\int_{t_0}^{t_f} \Big[&(\mathbf{\Psi}(t, t_0)\,\vec{x}(t_0))^T\mathbf{Q}(t)\,\mathbf{\Psi}(t, t_0)\,\vec{x}(t_0) \\ &+ (-\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\mathbf{P}(t)\,\mathbf{\Psi}(t, t_0)\,\vec{x}(t_0))^T\,\mathbf{R}(t) \\ &(-\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\mathbf{P}(t)\,\mathbf{\Psi}(t, t_0)\,\vec{x}(t_0))\Big]\,dt \\ &+ \frac{1}{2}\,(\mathbf{\Psi}(t_f, t_0)\,\vec{x}(t_0))^T\mathbf{P_1}\,(\mathbf{\Psi}(t_f, t_0)\,\vec{x}(t_0)). \end{aligned}$$

The (fixed) initial condition can be pulled out of the integral such that

$$\begin{aligned} V(t_0, \vec{x}_0) = \;& \frac{1}{2}\vec{x}(t_0)^T\bigg(\int_{t_0}^{t_f} \mathbf{\Psi}(t, t_0)^T\Big[\mathbf{Q}(t) \\ &+ \mathbf{P}(t)^T\mathbf{B}(t)\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T\mathbf{P}(t)\Big]\,\mathbf{\Psi}(t, t_0)\,dt \\ &+ \mathbf{\Psi}(t_f, t_0)^T\mathbf{P_1}\,\mathbf{\Psi}(t_f, t_0)\bigg)\,\vec{x}(t_0). \end{aligned} \tag{23}$$

Define the expression in the parenthetical above as $\mathbf{M}(t_0)$. It can be directly verified that this matrix is symmetric and positive semidefinite. Moreover, it is the integral expression for a corresponding matrix differential equation. Differentiation reveals its differential equation,

$$\begin{aligned} \frac{d}{d\tau}\mathbf{M}(\tau) = \;& \frac{d}{d\tau}\int_{\tau}^{t_f} \mathbf{\Psi}(t, \tau)^T\Big[\mathbf{Q}(t) + \mathbf{P}(t)^T\mathbf{B}(t)\mathbf{R}^{-1}(t) \\ &\mathbf{B}(t)^T\mathbf{P}(t)\Big]\mathbf{\Psi}(t, \tau)\,dt \\ &+ \frac{d}{d\tau}\mathbf{\Psi}(t_f, \tau)^T\mathbf{P_1}\,\mathbf{\Psi}(t_f, \tau) \\ =& -\Big(\mathbf{Q}(\tau) + \mathbf{P}(\tau)^T\mathbf{B}(\tau)\mathbf{R}^{-1}(\tau)\mathbf{B}(\tau)^T\mathbf{P}(\tau)\Big) \\ &- \mathbf{\Lambda}(\tau)^T\bigg(\int_{\tau}^{T} \mathbf{\Psi}(t, \tau)^T\Big[\mathbf{Q}(t) + \mathbf{P}(t)^T\mathbf{B}(t) \\ &\mathbf{R}^{-1}(t)\mathbf{B}(t)^T\mathbf{P}(t)\Big]\mathbf{\Psi}(t, \tau)\,dt\bigg) \\ &-\bigg(\int_{\tau}^{T} \mathbf{\Psi}(t, \tau)^T\Big[\mathbf{Q}(t) + \mathbf{P}(t)^T\mathbf{B}(t) \\ &\mathbf{R}^{-1}(t)\mathbf{B}(t)^T\mathbf{P}(t)\Big]\mathbf{\Psi}(t, \tau)\,dt\bigg)\mathbf{\Lambda}(\tau) \\ &- \mathbf{\Lambda}(\tau)^T\mathbf{\Psi}(t_f, \tau)^T\mathbf{P_1}\,\mathbf{\Psi}(t_f, \tau) \\ &- \mathbf{\Psi}(t_f, \tau)^T\mathbf{P_1}\,\mathbf{\Psi}(t_f, \tau)\mathbf{\Lambda}(\tau). \end{aligned} \tag{24}$$

Restated, (24) is equivalent to

$$\begin{aligned} \dot{\mathbf{M}}(\tau) =& -\mathbf{\Lambda}(\tau)^T\mathbf{M}(\tau) - \mathbf{M}(\tau)\mathbf{\Lambda}(\tau) \\ &- \mathbf{Q}(\tau) - \mathbf{P}(\tau)^T\mathbf{B}(\tau)\mathbf{R}^{-1}(\tau)\mathbf{B}(\tau)^T\mathbf{P}(\tau) \\ =& -\mathbf{A}(\tau)^T\mathbf{M}(\tau) + \mathbf{P}(\tau)\,\mathbf{B}(\tau)\,\mathbf{R}^{-1}(\tau)\,\mathbf{B}(\tau)^T\mathbf{M}(\tau) \\ &- \mathbf{M}(\tau)\mathbf{A}(\tau) + \mathbf{M}(\tau)\,\mathbf{B}(\tau)\,\mathbf{R}^{-1}(\tau)\,\mathbf{B}(\tau)^T\mathbf{P}(\tau) \\ &- \mathbf{Q}(\tau) - \mathbf{P}(\tau)^T\mathbf{B}(\tau)\mathbf{R}^{-1}(\tau)\mathbf{B}(\tau)^T\mathbf{P}(\tau), \end{aligned} \tag{25}$$

with terminal condition $\mathbf{M}(t_f) = \mathbf{P_1}$. In this form, the equation for $\dot{\mathbf{M}}(\tau)$ resembles Riccati equation (17). In fact, if $\mathbf{M}(\tau)$ in (25) is replaced with symmetric positive semidefinite solution $\mathbf{P}(\tau)$ to the Riccati equation, (25) becomes (17). Because solutions to the Riccati equation satisfy both differential equations (25) and (17) subject to the same terminal condition, the value function can be expressed in terms of $\mathbf{P}(\tau = t_0)$ as (19). ■

E. Existence and Uniqueness of Global Feedback Solutions

Mentioned earlier, solutions to Riccati equation (17) are only guaranteed to exist in a neighborhood of the final time based on continuity Assumption 2. It is only in this neighborhood where (3) results in an optimal value function of the form (19). To show that the optimal value function exists and is calculable over $[t_0, t_f]$, it is necessary to guarantee solutions to (17) exist over the horizon.

Lemma 6: Based on continuity Assumption 2, solutions to Riccati differential equation (17) exist and are unique $\forall t \in [t_0, t_f]$.

Proof: Applying the global extension of the Picard–Lindelöf theorem from [12], solutions to nonlinear differential equations

$$\dot{\vec{h}}(t) = \vec{g}(t, \vec{h}(t)) \tag{26}$$

with $\vec{h}(t_0) = \vec{h}_0$ exist and are unique over arbitrary horizons $[t_0, t_f]$ if $\vec{g}(\cdot, \vec{h}(t))$ is piecewise continuous in t, $\vec{g}(t, \cdot)$ is locally Lipschitz continuous in $\vec{h}(t)$, and solutions are bounded elements of the domain, $\vec{h}(t) \in \mathbb{R}^{n\times 1}$. As in [1], [15], this theorem generalizes to the backwards Riccati matrix differential equation with terminal cost.

Substituting $\vec{h}(t)$ for $\mathbf{P}(t)$, initial condition $\vec{h}_0$ for terminal condition $\mathbf{P_1}$, and $\vec{g}$ for $\dot{\mathbf{P}}$, one can show (17) is locally Lipschitz continuous in $\mathbf{P}(t)$ by proving that $\frac{\partial \mathbf{g}(t,\cdot)}{\partial \mathbf{P}(t)}$ is uniformly continuous in $\mathbf{P}(t)$ for all fixed t (see [12]). Based on the partial derivative,

$$\frac{\partial \mathbf{g}(t, \cdot)}{\partial \mathbf{P}(t)} = -\mathbf{A}(t) - \mathbf{A}(t)^T + \mathbf{P}(t)\,\mathbf{B}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{B}(t)^T, \tag{27}$$

it should be clear that these conditions are met. Further, it follows directly from Assumption 2 that $\dot{\mathbf{P}}(t) \in \widetilde{\mathcal{C}}^0$ with respect to t. It therefore suffices to show that solutions to (17) cannot exhibit finite escape time. A simple proof of this that relies only on the positive semidefinite form of optimal value function (19) and the $\mathcal{C}^0$ integral expression for $\mathbf{P}(t)$ (see expression for $\mathbf{M}(t)$) is included in [1], [15]. Based on these results, one can conclude the Riccati equation exists and is unique for $\forall t \in [t_0, t_f]$. ■

In summary, Assumptions 1 and 2 guarantee a continuous, unique symmetric and positive semidefinite matrix, $\mathbf{P}(t)$, exists. This matrix sets the first variation $\delta J = 0$ continuously, for all $\widetilde{\mathcal{C}}^0$ control perturbations when $\vec{u}(t)$ is defined by (3). Extremal controls therefore exist and are at least

$\widetilde{\mathcal{C}}^0$ $\forall t \in [t_0, t_f]$. As the optimal value function is convex, this is necessary and sufficient to show controls (3) are globally optimal over arbitrary time horizon $[t_0, t_f]$ and completes the proof of Theorem 1.

III. Discussion and Conclusions

The derivation provided in Section II is organized very differently than most others. One popular approach in [1] begins by proving cost (2) must be representable as a positive semidefinite quadratic form (that of the optimal value function (19)). After application of (1), (2), and a completions of squares argument, the authors show the HJB equations result in

$$\begin{aligned}\frac{\partial V(t,\vec{x})}{\partial t} &= \vec{x}^T\dot{\mathbf{P}}\vec{x}\\ &= -\min_{\vec{u}}\,[(\vec{u}+\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\vec{x})^T\mathbf{R}(\vec{u}+\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\vec{x})\\ &\qquad + \vec{x}^T(\mathbf{Q}-\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\mathbf{A}^T\mathbf{P})\vec{x}]\,,\end{aligned}$$

where time dependencies have been dropped for brevity and $V(t_f, \vec{x}(t_f)) = \vec{x}(t_f)^T\mathbf{P_1}\,\vec{x}(t_f)$. In performing the minimization, they solve for the optimal regulator and arrive at a differential Riccati equation (17) that must be positive semi-definite because of its role in the optimal value function. However, the minimization requires temporary additional regularity assumptions (quantities in Assumption 2 must be $\mathcal{C}^2$). The approach also obscures the meaning of the Riccati equation as a linear map between the adjoint variable and the state required to translate between optimal feedforward expression (11) and LQR feedback solutions (3).

Also, as mentioned earlier, methods that apply EL equations and/or PMP usually solve the LQR problem by constrained optimization where perturbations are applied to the state (and constrained to zero at the initial time to enforce the initial condition $\vec{x}(t_0) = \vec{x}_0$). This fixed-time variable-endpoint formulation is equivalent to maximizing Hamiltonian system,

$$H \triangleq \vec{\rho}(t)^T(\mathbf{A}(t)\vec{x}(t)+\mathbf{B}(t)\vec{u}(t)) - l(t,\vec{x}(t),\dot{\vec{x}}(t)). \quad (28)$$

In this Hamiltonian setting, $\vec{\rho}(t)$ represents a generalized momentum vector, which PMP confirms as satisfying adjoint equation (15). In the EL interpretation, $\vec{\rho}(t)$ is related to the vector of Lagrange multipliers that enforce dynamic constraints (1). In these formulations, the terminal condition on the adjoint is necessary to accommodate the free endpoint perturbation and ensure the optimization problem is well-posed [15]. In the unconstrained optimization setting from this paper, it is noteworthy that the same variable shows up for very different reasons. Here, $\vec{\rho}(t)$ is simply part of an affine term in the first variation of the cost (10) that does not depend on $\vec{u}(t)$. It is calculated from (15) to avoid explicit computation of a state transition matrix.

In contrast to these alternatives, the derivation presented in this paper is constructive. It builds from basic concepts of variational calculus and shows how critical terms from LQR theory such as the Riccati equation and adjoint show up regardless of problem formulation. Though the methods applied are specific to the LQR problem, they prove the LQR solution generalizes to systems where the dynamics and cost are piecewise continuous in time. Thus, the derivation is valid for a larger array of systems than classical LQR derivations based on HJB or the EL equations.

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