Structured Linearization of Discrete Mechanical Systems on Lie Groups: a Synthesis of Analysis and Control

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***Abstract*— Lie group variational integrators have the advantages of both variational and Lie group integrators, which preserve the momentum, symplectic form, holonomic constraints and the Lie group structure. In addition, their long-time energy stable behaviour and coordinate-independent nature make it quite suitable to simulate a variety of mechanical systems. The structure-preservation of a Lie group variational integrator implies its linearization is structure-preserving as well, thus we call such a linearization "structured linearization". However, due to the implicit nature of variational integrators and the non-trivial differential structure of Lie groups, the structured linearization of Lie group variational integrators is much more complicated than that in generalized coordinates. In this paper, we formulate the structured linearization of Lie group variational integrators to synthesize existing analysis and control tools. To illustrate the utility of the paper, LQR controllers are constructed directly on constrained Lie groups for the asymmetric 3D pendulum and quadrotor with a suspended load, simulation results show that both controllers have a large basin of attraction.**

***Index-Terms*— structured linearization; Lie groups; variational integrators; LQR.**

I. INTRODUCTION

Different from the usual approach to simulate mechanical systems, which first derives continuous dynamic equations through the Lagrange-d'Alembert principle and then discretizes the system to yield a numerical integrator, *variational integrators* approximates the action integral discretely and employ the discrete Lagrange-d'Alembert principle to obtain the update rule for the discrete trajectory [1], [2]. Compared with other numerical integrators, variational integrators preserve the momentum, symplectic form and holonomic constraints [1]. Most importantly, though the energy is not preserved, the variational integrator demonstrates the behavior of long-time energy stability, which is very important to simulate complex systems [3]–[5]. Recently increasing attention has been paid to Lie group variational integrators [6]–[11]. Besides preserving motion invariants, Lie group variational integrators also preserve the Lie group structure and do not have the problem of singularity [10].

Linearization is one of the most important and frequently-used techniques in both analysis and control, such as optimization, stability analysis, LQR regulators etc.. One of the things that interest us most is the linearization of variational integrators. Due to the structure-preserving properties, the linearization of variational integrators, in particular Lie group variational integrators, remains structure-preserving, which we call "*structured linearization*". Though structured linearization of variational integrators in generalized coordinates has been studied in [12], little work has been done to the structured linearization of Lie group variational integrators. In fact, the structured linearization of variational integrators on non-abelian Lie groups is much more complicated due to the non-trivial differential structure. As a comparison, linearization of the other Lie group integrators is relatively trivial.

Continuous-time linearization of Lie group dynamical system has been studied in [13] through perturbation theory, which derives and then linearizes the control-input-perturbed trajectory in exponential coordinates. Another way to derive continuous-time linearization is to linearize the approximate discrete-time system and take limits of the resulting linearized dynamics as the time step approaches zero.

The main contribution of this paper is the derivation of structured linearization of discrete mechanical systems on Lie groups, including constraints, and compared with other (structured) linearization techniques in generalized coordinates, there is *no need* to do chart switching to avoid singularity. The secondary contribution is to illustrate how structured linearization enables one to use classical control synthesis for vector spaces directly on Lie groups, even when the Lie group is non-abelian. We illustrate these results with a demonstration of classical *LQR theory* applied to examples.

The organization of the paper is as following: Section II provides some background information for Lie group variational integrators. Section III derives the structured linearization and the exact expressions of derivative terms used in structured linearization. In Section IV LQR controllers for two examples, the *asymmetric 3D pendulum* and the *quadrotor with a suspended load*, are presented, which demonstrates the utility of the paper.

II. LIE GROUP VARIATIONAL INTEGRATORS

A. Preliminaries

Instead of focusing on a certain type of motion group, such as $\mathbb{R}^3$, $SO(3)$ or $SE(3)$, we'll represent the n-dimensional configuration space Q as the *direct product* $G_1 \times G_2 \times \cdots \times G_n$ of *motion groups* G_i. Nearly all finite-dimensional mechanical systems can be depicted in this way and it will be particularly useful for mechanical systems comprised of interconnected rigid bodies. Moreover, the configuration space Q itself can also be regarded as a Lie group (more definitely, matrix Lie group), whose topological, geometric and Lie group properties can be obtained exactly from each G_i through the direct product representation. In addition,

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T_eQ, i.e. the Lie algebra of Q, is the direct sum $T_eG_1 \oplus T_eG_2 \oplus \cdots \oplus T_eG_n$ of Lie algebra T_eG_i for each G_i. And if a basis $\mathfrak{B}_i = \{E_i^1, E_i^2, \cdots E_i^{a_i}\}$ is specified to each T_eG_i ($\dim G_i = a_i$), the disjoint union $\mathfrak{B} = \coprod_i \mathfrak{B}_i$ forms a basis for T_eQ as well. The dual space T_e^*Q of T_eQ and its basis can also be defined in a similar way.

Given a smooth map $F: Q \longrightarrow M$ from Lie group Q to smooth manifold M, we can define a linear operator $\nabla_q F\big|_q : T_eQ \longrightarrow T_{F(q)}M$ such that

$$\nabla_q F\big|_q \cdot \eta = \frac{d}{ds} F(q \cdot \exp(s\eta)) \in T_{F(q)}M$$

for all $\eta \in T_eG$. If corresponding bases are specified for T_eG and $T_{F(q)}M$, then $\nabla_q F|_q$ can be represented as a Jacobian matrix. It could be found later that $\nabla_q F|_q$ is the most basic operator for structured linearization in both derivation and numerical computation. For brevity, the dependence $|_q$ will be dropped and $\nabla_q F|_q$ will be simply written as $\nabla_q F$.

In this paper, we assume readers have a basic knowledge of Lie group theory, which can be found in a variety of textbooks, e.g. [14].

B. The Continuous Lie Group Dynamics

There is an abundance of literature on *Lie group dynamics*, such as [15]–[17]. Here we only give a brief introduction to provide some background information.

In Lie group context, the Lagrangian on the configuration space Q is a $\mathbb{R}$-valued function $L : Q \times T_eQ \longrightarrow \mathbb{R}$ such that given $(g, \xi) = (g, g^{-1}\dot{g}) \in Q \times T_eQ$, the Lagrangian $L(g, \xi)$ is defined to be

$$L(g, \xi) := K(g, \xi) - V(g)$$

where $K(g, \xi)$ is the kinematic energy and $V(g)$ is the potential energy.

Given a generalized control force $f \in T_e^*Q$, the Lagrange-d'Alembert Principle requires that

$$\int_0^T \langle \nabla_g L, \eta\rangle + \langle \nabla_\xi L, \delta\xi\rangle \, dt + \int_0^T \langle f, \eta\rangle \, dt = 0$$

where $\eta = g^{-1}\delta g \in T_eG$ and $\langle, \rangle$ is the innerproduct pairing vector and dual-vector. It is not hard to derive the relation of $\delta\xi$ and η:

$$\delta\xi = \dot{\eta} + \mathrm{ad}_\xi \eta. \tag{1}$$

Hence we have

$$\frac{d}{dt}\nabla_\xi L - \mathrm{ad}_\xi^* \nabla_\xi L = \nabla_g L + f \tag{2}$$

where $\mathrm{ad}_\xi \eta = \xi\eta - \eta\xi$ and $\mathrm{ad}_\xi^* : T_e^*Q \longrightarrow T_e^*Q$ is a linear operator such that $\langle \mathrm{ad}_\xi^* \mu, \eta\rangle = \langle \mu, \mathrm{ad}_\xi \eta\rangle$ for any $\xi \in T_eQ$ and $\mu \in T_e^*Q$.

One thing we should mention is that these equations may fail to hold if Q is not a matrix Lie group.

C. The Variational Integrator on Lie Groups

Instead of discretizing continuous equations of motion, which is the case for a number of numerical integrators, e.g. Runge-Kutta method, the variational integrator is derived through the discretization of the Lagrange-d'Alembert Principle [1], [2], [18], [19].

Like variational integrator in generalized coordinates, the trajectory on Lie group is discretized through certain approximation

$$L_d(g_k, g_{k+1}) \approx \int_{k\Delta t}^{(k+1)\Delta t} L(g, \xi)\, dt.$$

To do this is, we can find a map $\tau : T_eQ \longrightarrow Q$ so that τ is invertible around the origin and the tangent map at the origin is the identity map. If such a τ exists, the "average velocity" can be estimated as $\xi_k \approx \frac{\tau^{-1}(g_k{}^{-1}g_{k+1})}{\Delta t}$ and the action integral $\int_{k\Delta t}^{(k+1)\Delta t} L(g, \xi)\, dt$ can be approximated through the trapezoid rule

$$L_d(g_k, g_{k+1}) := \frac{\Delta t}{2}[\underbrace{L(g_k, \xi_k)}_{\substack{\shortparallel \\ L_1}} + \underbrace{L(g_{k+1}, \xi_k)}_{\substack{\shortparallel \\ L_2}}]. \tag{3}$$

In fact, we can always find such a map τ and a more formal definition is given as follows [6].

Definition II.1. [6] The *retraction map* $\tau : T_eQ \longrightarrow Q$ on a Lie group Q is a C^2- diffeomorphism around the identity such that $\tau(\mathbf{0}) = \mathrm{I}$ and the tangent map at the origin $\mathrm{D}\tau|_\mathbf{0} : T_eQ \longrightarrow T_eQ$ is the identity map[1].

It can be easily checked the *exponential map* $\exp : T_eQ \longrightarrow Q$ is a retraction map. Another choice is the *Cayley map* $\mathrm{cay} : T_eQ \longrightarrow Q$ where

$$\mathrm{cay}(\xi) = (\mathrm{I} - \frac{\xi}{2})^{-1}(\mathrm{I} + \frac{\xi}{2}).$$

Before applying the discrete Lagrange-d'Alembert Principle, we also ought to relate the velocity variation $\delta\xi_k \in T_eQ$ with the configuration variations $\eta_k, \eta_{k+1} \in T_eQ$, where $\eta_k = g_k^{-1}\delta g_k$, as Eq. (1) in continuous settings. However, this is not trivial and some operators are needed to derive this relation.

Definition II.2. [6]–[8] Given a retraction map $\tau : T_eQ \longrightarrow Q$ on Lie group Q, the *right trivialized tangent* $\mathrm{d}\tau_\xi : T_eQ \longrightarrow T_eQ$ and the *right trivialized tangent inverse* $\mathrm{d}\tau_\xi^{-1} : T_eQ \longrightarrow T_eQ$ are defined to be linear operators such that for any $\eta, \eta' \in T_eQ$ we have

$$\mathrm{d}\tau_\xi|_\eta \cdot \eta' \cdot \tau(\eta) = \nabla_\xi \tau|_\eta \cdot \eta' \tag{4}$$

$$\mathrm{d}\tau_\xi^{-1}|_\eta \cdot (\eta' \cdot \tau(-\eta)) = \nabla_\xi \tau^{-1}|_\eta \cdot \eta'. \tag{5}$$

For the exponential map, we have [20]

$$\mathrm{d}\exp_x y = \sum_{j=0}^\infty \frac{1}{(j+1)!}\mathrm{ad}_x^j y$$

[1]The requirement that $\mathrm{D}\tau|_\mathbf{0}$ is the identity map is important because it ensures $\dfrac{K(g_k, \xi_k) + K(g_{k+1}, \xi_k)}{2}\Delta t \approx \int_{k\Delta t}^{(k+1)\Delta t} K(g, \xi)\, dt$.

$$\mathrm{d}\exp_x^{-1} y = \sum_{j=0}^{\infty} \frac{B_j}{j!}\mathrm{ad}_x^j y$$

where B_j are Bernoulli numbers and for the Cayley map

$$\mathrm{dcay}_x y = (\mathrm{I} - \frac{x}{2})^{-1} y (\mathrm{I} + \frac{x}{2})^{-1}$$

$$\mathrm{dcay}_x^{-1} y = (\mathrm{I} - \frac{x}{2}) y (\mathrm{I} + \frac{x}{2}).$$

Note if the corresponding Lie group is abelian, then $\mathrm{d}\exp_x$, $\mathrm{d}\exp_x^{-1}$, dcay_x and dcay_x^{-1} are all identity maps, which is the case in coordinates.

Eqs. (4) and (5) are very useful, in particular Eq. (5) and its derivatives will be frequently used in derivation of structured linearization on Lie groups in this paper.

By taking derivatives on $\xi_k \approx \frac{\tau^{-1}(g_k{}^{-1} g_{k+1})}{\Delta t}$ and then applying Eq. (5), the velocity variation and configuration variation can be related as follows

$$\delta\xi_k = \frac{1}{\Delta t} \cdot \mathrm{d}\tau_{\xi\Delta t}^{-1}(-\eta_k + \mathrm{Ad}_{\tau(\xi\Delta t)}\eta_{k+1}) \tag{6}$$

where $\mathrm{Ad}_g : T_eQ \longrightarrow T_eQ$ is a linear operator such that $\mathrm{Ad}_g\xi = g\xi g^{-1}$ for any $g \in Q$ and $\xi \in T_eQ$, .

Now we can apply the discrete Lagrange-d'Alembert Principle to derive the update rule for Lie group variational integrators. If the system is unforced, we have

$$\delta \sum_{k=0}^{N-1} L_d(g_k, g_{k+1}) = \delta \sum_{k=0}^{N-1} \frac{L(g_k, \xi_k) + L(g_{k+1}, \xi_k)}{2} = 0,$$

hence

$$D_2L_d(g_{k-1}, g_k) + D_1L_d(g_k, g_{k+1}) = 0 \tag{7}$$

where

$$D_1L_d(g_k, g_{k+1}) = \frac{\Delta t}{2} \cdot \nabla_g L_1 - \frac{1}{2}\mathrm{d}\tau_{\xi\Delta t}^{-1\,*} \cdot (\nabla_\xi L_1 + \nabla_\xi L_2)$$

$$D_2L_d(g_k, g_{k+1}) = \frac{\Delta t}{2} \cdot \nabla_g L_2 + \frac{1}{2}\mathrm{Ad}^*_{\tau(\xi\Delta t)} \cdot \mathrm{d}\tau_{\xi\Delta t}^{-1\,*} \cdot (\nabla_\xi L_1 + \nabla_\xi L_2).$$

The star operators $(\#)^*$ appear in D_1L_d and D_2L_d is defined in the same way as ad_ξ^* in Eq. (2).

In this paper we prefer to rewrite Eq. (7) in an equivalent *position-momentum* form only depending on the current and future time steps, so we have a one-step map, which results the update rule

$$p_k + D_1L_d(g_k, g_{k+1}) = 0 \tag{8a}$$

$$p_{k+1} = D_2L_d(g_k, g_{k+1}) \tag{8b}$$

where p_k, $D_1L_d(g_k, g_{k+1})$, $D_2L_d(g_k, g_{k+1}) \in T_e^*Q$.

Taking the same approach, we can also get the update rules of Lie group variational integrators for forced systems

$$p_k + D_1L_d(g_k, g_{k+1}) + F_{k+1}^- = 0 \tag{9a}$$

$$p_{k+1} = D_2L_d(g_k, g_{k+1}) + F_{k+1}^+ \tag{9b}$$

and constrained forced systems

$$p_k + D_1L_d(g_k, g_{k+1}) + F_{k+1}^- - Dh^T(g_k)\lambda_k = 0 \tag{10a}$$

$$h(g_{k+1}) = 0 \tag{10b}$$

$$p_{k+1} = D_2L_d(g_k, g_{k+1}) + F_{k+1}^+. \tag{10c}$$

Eqs. (9) and (10) will be used to derive structured linearization in Section III.

D. Conclusion

In this section, we reviewed the continuous Lie group dynamics and derived the update rules for Lie group variational integrators. We also introduced the idea of retraction map, right trivialized tangent and the right trivialized tangent inverse. The right trivialized tangent inverse will be frequently used in the derivation of the structured linearization. For more details about Lie group variational integrators as well as other approaches to derive the update rule, interested readers can refer to [6]–[11].

III. STRUCTURED LINEARIZATION

A. An Application of Right Trivialized Tangent Inverse

With the help of the right trivialized tangent inverse, we successfully relate the velocity and configuration variations. In fact, we can go much further than that, which the following proposition indicates.

Proposition 1. Given a matrix Lie group and a C^2 map $F_d(g_k, g_{k+1}) : G \times G \longrightarrow \mathbb{R}^n$, if there exists another C^2 map $F(g_k, g_{k+1}, \xi) : G \times G \times T_eG \longrightarrow \mathbb{R}^n$, a retraction map $\tau : T_eG \longrightarrow G$ and $\Delta t \in \mathbb{R}^+$ satisfying[2]

$$F_d(g_k, g_{k+1}) = F(g_k, g_{k+1}, \frac{1}{\Delta t}\tau^{-1}(g_k^{-1} g_{k+1})),$$

then there exists $\mathbb{R}^n$-valued forms D_1F_d and D_2F_d on T_eG

$$D_1F_d = \nabla_{g_k}F - \frac{1}{\Delta t} \cdot \mathrm{d}\tau_{\xi\Delta t}^{-1\,*} \circ \nabla_\xi F \tag{11a}$$

$$D_2F_d = \nabla_{g_{k+1}}F + \frac{1}{\Delta t} \cdot \mathrm{Ad}^*_{\tau(\xi\Delta t)} \circ \mathrm{d}\tau_{\xi\Delta t}^{-1\,*} \circ \nabla_\xi F. \tag{11b}$$

such that $\delta F_d = \langle D_1F_d, \eta_k\rangle + \langle D_2F_d, \eta_{k+1}\rangle$.

The proof of Proposition 1 is simply to take variation and then apply Eq. (6).

In Eq. (11) linear operators $\nabla_\# F$ are essentially $\mathbb{R}^n$-valued forms on T_eG, i.e. linear maps from Lie algebra T_eG to $\mathbb{R}^n$.

With Proposition 1 it is possible for us to continue taking derivatives on these update rules and derive the structured linearization.

B. Derivation of Structured Linearization on Lie Groups

Here we will work on the structured linearization of the update rules for forced systems (Eqs. (9a) and (9b)).

To make our derivation fully make sense, we need to first clarify the meaning and interpretation of each term in Eqs. (9a) and (9b). Though the derivation of Eq. (9) implies all these terms, i.e. p_k, $D_1L_d(g_k, g_{k+1})$, F_{k+1}^- etc., are $\mathbb{R}$-valued forms on T_eQ (elements in T_e^*Q), we'll interpret them as $\mathbb{R}^n$-valued functions to apply Proposition 1. This

[2]We suppose $\tau^{-1}(g_k^{-1} g_{k+1})$ always exists.

interpretation doesn't lose any generality since as long as the expression of these $\mathbb{R}^n$-valued functions under one basis of T_eG is known, their expressions under other bases will be uniquely determined through basis transformation matrices.

For simplicity, we'll denote $L_d(g_k, g_{k+1})$ as L_{k+1}.

By Proposition 1, we take variation of Eq. (9a)

$$\begin{aligned} \langle \mathrm{I}, \delta p_k\rangle + \langle D_1D_1L_{k+1} + D_1F^-_{k+1}, \eta_k\rangle + \\ \langle D_2D_1L_{k+1} + D_2F^-_{k+1}, \eta_{k+1}\rangle + \\ \langle D_3F^-_{k+1}, \delta u_k\rangle = 0. \end{aligned} \tag{12}$$

Based on our interpretation of terms in Eqs. (9a) and (9b), terms like $D_1D_1L_{k+1}$, $D_2D_1L_{k+1}$, $D_1F^-_{k+1}$ in Eq. (12) are $\mathbb{R}^n$-valued forms on T_eG.

Without loss of generality, let $M_{k+1} = D_2D_1L_{k+1} + D_2F^-_{k+1}$ and $\eta_{k+1} = \frac{\partial g_{k+1}}{\partial g_k}\eta_k + \frac{\partial g_{k+1}}{\partial p_k} + \frac{\partial g_{k+1}}{\partial u_k}\delta u_k$, where $\frac{\partial g_{k+1}}{\partial g_k}$, $\frac{\partial g_{k+1}}{\partial p_k}$ and $\frac{\partial g_{k+1}}{\partial u_k}$ are tangent maps to T_eQ, we can write Eq. (12) as

$$\begin{aligned} &\langle \mathrm{I} + \left(\tfrac{\partial g_{k+1}}{\partial p_k}\right)^* \circ M_{k+1}, \delta p_k\rangle + \\ &\langle D_1D_1L_{k+1} + D_1F^-_{k+1} + \left(\tfrac{\partial g_{k+1}}{\partial g_k}\right)^* \circ M_{k+1}, \eta_k\rangle + \\ &\langle D_3F^-_{k+1} + \left(\tfrac{\partial g_{k+1}}{\partial u_k}\right)^* \circ M_{k+1}, \delta u_k\rangle = 0 \end{aligned}$$

which is equivalent to

$$\left(\tfrac{\partial g_{k+1}}{\partial g_k}\right)^* \circ M_{k+1} + D_1D_1L_{k+1} + D_1F^-_{k+1} = 0 \tag{13a}$$

$$\left(\tfrac{\partial g_{k+1}}{\partial p_k}\right)^* \circ M_{k+1} + \mathrm{I} = 0 \tag{13b}$$

$$\left(\tfrac{\partial g_{k+1}}{\partial u_k}\right)^* \circ M_{k+1} + D_3F^-_{k+1} = 0. \tag{13c}$$

If M_{k+1} is non-singular, $\frac{\partial g_{k+1}}{\partial g_k}$, $\frac{\partial g_{k+1}}{\partial p_k}$ and $\frac{\partial g_{k+1}}{\partial p_k}$ is well defined from Eq. (13).

To make Eq. (13) computable, bases for corresponding vector spaces should be specified. Here we adopt the convention in differential geometry, where elements in T_eQ are column vectors, elements in T_e^*Q are row vectors and $\mathbb{R}^m$-valued forms on n-dimensional vector spaces are $m \times n$ matrices. Then Eq. (13) can be rewritten as

$$M_{k+1} \cdot \tfrac{\partial g_{k+1}}{\partial g_k} + D_1D_1L_{k+1} + D_1F^-_{k+1} = 0$$

$$M_{k+1} \cdot \tfrac{\partial g_{k+1}}{\partial p_k} + \mathrm{I} = 0$$

$$M_{k+1} \cdot \tfrac{\partial g_{k+1}}{\partial u_k} + D_3F^-_{k+1} = 0.$$

If $M_{k+1} = D_2D_1L_{k+1} + D_2F^-_{k+1}$ is invertible, we have

$$\tfrac{\partial g_{k+1}}{\partial g_k} = -M^{-1}_{k+1} \cdot [D_1D_1L_{k+1} + D_1F^-_{k+1}] \tag{15a}$$

$$\tfrac{\partial g_{k+1}}{\partial p_k} = -M^{-1}_{k+1} \tag{15b}$$

$$\tfrac{\partial g_{k+1}}{\partial u_k} = -M^{-1}_{k+1} \cdot D_3F^-_{k+1}. \tag{15c}$$

Taking the same approach to Eq. (9b) and substituting Eq. (15), we can get $\frac{\partial p_{k+1}}{\partial g_k}$, $\frac{\partial p_{k+1}}{\partial p_k}$ and $\frac{\partial p_{k+1}}{\partial u_k}$

$$\begin{aligned} \tfrac{\partial p_{k+1}}{\partial g_k} = D_1D_2L_{k+1} + D_1F^+_{k+1} + \\ [D_2D_2L_{k+1} + D_2F^+_{k+1}] \cdot \tfrac{\partial g_{k+1}}{\partial g_k} \end{aligned} \tag{16a}$$

$$\tfrac{\partial p_{k+1}}{\partial p_k} = [D_2D_2L_{k+1} + D_2F^+_{k+1}] \cdot \tfrac{\partial g_{k+1}}{\partial p_k} \tag{16b}$$

$$\tfrac{\partial p_{k+1}}{\partial u_k} = D_3F^+_{k+1} + [D_2D_2L_{k+1} + D_2F^+_{k+1}] \cdot \tfrac{\partial g_{k+1}}{\partial u_k}. \tag{16c}$$

Eqs. (15) and (16) are exactly the structured linearization for forced systems, which will be used to design LQR controllers in Section IV.

C. Derivatives Computation

Though Eqs. (15) and (16) look the same as structured linearization of mechanical systems in generalized coordinates [12], the computation of derivatives, such as $D_1D_1L_d$, $D_2D_1L_d$, $D_2D_2L_d$, are not so trivial if Q is a non-abelian Lie group.

If the trapezoid rule (Eq. (3)) is used, the first derivatives are

$$\begin{aligned} &D_1L_d(g_k, g_{k+1}) \\ =&\frac{\Delta t}{2} \cdot \nabla_g L_1 - \frac{1}{2} d\tau^{-T}_{\xi\Delta t} \cdot (\nabla_\xi L_1 + \nabla_\xi L_2) \end{aligned}$$

$$\begin{aligned} &D_2L_d(g_k, g_{k+1}) \\ =&\frac{\Delta t}{2} \cdot \nabla_g L_2 + \frac{1}{2} \mathrm{Ad}^T_{\tau(\xi\Delta t)} \cdot d\tau^{-T}_{\xi\Delta t} \cdot (\nabla_\xi L_1 + \nabla_\xi L_2). \end{aligned}$$

For second derivatives $D_{(\cdot)}D_{(*)}L_d$, we have

$$\begin{aligned} &D_1D_1L_d(g_k, g_{k+1}) \\ =&\frac{\Delta t}{2}\nabla^2_g L_1 - \frac{1}{2}\mathrm{d}\tau^{-T}_{\xi\Delta t} \cdot \nabla_g\nabla_\xi L_1 - \frac{1}{2}\nabla_\xi\nabla_g L_1 \cdot \mathrm{d}\tau^{-1}_{\xi\Delta t} + \\ &\frac{1}{2\Delta t}\Big[\mathrm{d}\tau^{-T}_{\xi\Delta t} \cdot (\nabla^2_\xi L_1 + \nabla^2_\xi L_2) + \\ &(\nabla^T_\xi L_1 \otimes I + \nabla^T_\xi L_2 \otimes I) \cdot \partial_\xi(\mathrm{d}\tau^{-T}_{\xi\Delta t})^S\Big] \cdot \mathrm{d}\tau^{-1}_{\xi\Delta t} \end{aligned}$$

$$\begin{aligned} &D_2D_1L_d(g_k, g_{k+1}) \\ =&-\frac{1}{2}\mathrm{d}\tau^{-T}_{\xi\Delta t} \cdot \nabla_g\nabla_\xi L_2 + \frac{1}{2}\nabla_\xi\nabla_g L_1 \cdot \mathrm{d}\tau^{-1}_{\xi\Delta t} \cdot \mathrm{Ad}_{\xi\Delta t} - \\ &\frac{1}{2\Delta t}\Big[\mathrm{d}\tau^{-T}_{\xi\Delta t} \cdot (\nabla^2_\xi L_1 + \nabla^2_\xi L_2) + \\ &(\nabla^T_\xi L_1 \otimes I + \nabla^T_\xi L_2 \otimes I) \cdot \partial_\xi(\mathrm{d}\tau^{-T}_{\xi\Delta t})^S\Big] \cdot \mathrm{d}\tau^{-1}_{\xi\Delta t} \cdot \mathrm{Ad}_{\xi\Delta t} \end{aligned}$$

$$D_1D_2L_d(g_k, g_{k+1}) = D_2D_1L_d(g_k, g_{k+1})^T$$

$$\begin{aligned} &D_2D_2L_d(g_k, g_{k+1}) \\ =&\frac{\Delta t}{2}\nabla^2_g L_2 + \frac{1}{2}\mathrm{Ad}^T_{\tau(\xi\Delta t)} \cdot \mathrm{d}\tau^{-T}_{\xi\Delta t} \cdot \nabla_g\nabla_\xi L_2 + \\ &\frac{1}{2}\nabla_\xi\nabla_g L_2 \cdot \mathrm{d}\tau^{-1}_{\xi\Delta t} \cdot \mathrm{Ad}_{\tau(\xi\Delta t)} + \\ &\frac{1}{2\Delta t}\Big[\mathrm{Ad}^T_{\tau(\xi\Delta t)} \cdot \mathrm{d}\tau^{-T}_{\xi\Delta t} \cdot (\nabla^2_\xi L_1 + \nabla^2_\xi L_2) + \\ &(\nabla^T_\xi L_1 \otimes I + \nabla^T_\xi L_2 \otimes I) \cdot \partial_\xi(\mathrm{Ad}^T_{\tau(\xi\Delta t)} \cdot \mathrm{d}\tau^{-T}_{\xi\Delta t})^S\Big] \cdot \\ &\mathrm{d}\tau^{-1}_{\xi\Delta t} \cdot \mathrm{Ad}_{\tau(\xi\Delta t)} \end{aligned}$$

where "$\otimes$" is the Kronecker product and "$(\cdot)^S$" is the stack operator [21]. If Q is abelian, these derivatives will be reduced to the case of multi-variable calculus.

Note even if the commutativity of Lie algebra elements in the same T_eG_i may not hold, the Lie algebra elements in T_eG_i is always commutative with these in T_eG_j if $i \neq j$. This means the computational process can be simplified by computing the derivatives in terms of blocks.

Let blocks of derivatives for Lagrangian $L : Q \times T_eQ \longrightarrow \mathbb{R}$ associated with G_i and G_j be denoted as follows

$$\nabla_{(\cdot)_i} L = \begin{bmatrix} \nabla_{(\cdot)_i^1} L & (\nabla_{(\cdot)_i^2} L & \cdots & \nabla_{(\cdot)_i^{a_i}} L \end{bmatrix}^T_{a_i \times 1}$$

$$\nabla_{(\cdot)_i} \nabla_{(\#)_j} L = \begin{bmatrix} \nabla_{(\cdot)_i^1} \nabla_{(\#)_j^1} L & \cdots & \nabla_{(\cdot)_i^{a_i}} \nabla_{(\#)_j^1} L \\ \vdots & \ddots & \vdots \\ \nabla_{(\cdot)_i^1} \nabla_{(\#)_j^{a_j}} L & \cdots & \nabla_{(\cdot)_i^{a_i}} \nabla_{(\#)_j^{a_j}} L \end{bmatrix}_{a_j \times a_i}$$

where "$\cdot$" or "$\#$" are either E as derivatives taken in G_i[3] or ξ as derivatives taken in T_eG_i. Then we have

$$\begin{aligned} & [\, D_1 L_d(g_k, g_{k+1}) \,]_i \\ =& \frac{\Delta t}{2} \nabla_{E_i} L_1 - \frac{1}{2} d\tau^{-T}_{\xi_i \Delta t} \cdot (\nabla_{\xi_i} L_1 + \nabla_{\xi_i} L_2) \end{aligned}$$

$$\begin{aligned} & [\, D_2 L_d(g_k, g_{k+1}) \,]_i \\ =& \frac{\Delta t}{2} \nabla_{E_i} L_2 + \frac{1}{2} \mathrm{Ad}^T_{\tau(\xi_i \Delta t)} \cdot d\tau^{-T}_{\xi_i \Delta t} \cdot (\nabla_{\xi_i} L_1 + \nabla_{\xi_i} L_2). \end{aligned}$$

And for second derivatives

$$\begin{aligned} & [D_1 D_1 L_d(g_k, g_{k+1})]_{i,j} \\ =& \frac{\Delta t}{2} \nabla_{E_i} \nabla_{E_j} L_1 - \frac{1}{2} d\tau^{-T}_{\xi_j \Delta t} \cdot \nabla_{E_i} \nabla_{\xi_j} L_1 - \\ & \frac{1}{2} \nabla_{\xi_i} \nabla_{E_j} L_1 \cdot d\tau^{-1}_{\xi_i \Delta t} + \\ & \frac{1}{2\Delta t} d\tau^{-T}_{\xi_j \Delta t} \cdot (\nabla_{\xi_i} \nabla_{\xi_j} L_1 + \nabla_{\xi_i} \nabla_{\xi_j} L_2) \cdot d\tau^{-1}_{\xi_i \Delta t} + \\ & \frac{\delta_{i,j}}{2\Delta t} (\nabla^T_{\xi_i} L_1 \otimes I + \nabla^T_{\xi_i} L_2 \otimes I) \cdot \partial_{\xi_i} (d\tau^{-T}_{\xi_i \Delta t})^S \cdot d\tau^{-1}_{\xi_i \Delta t} \end{aligned}$$

$$\begin{aligned} & [D_2 D_1 L_d(g_k, g_{k+1})]_{i,j} \\ =& -\frac{1}{2} \mathrm{d}\tau^{-T}_{\xi_j \Delta t} \cdot \nabla_{E_i} \nabla_{\xi_j} L_2 + \frac{1}{2} \nabla_{\xi_i} \nabla_{E_j} L_1 \cdot \mathrm{d}\tau^{-1}_{\xi_i \Delta t} \cdot \mathrm{Ad}_{\xi_i \Delta t} \\ & - \frac{1}{2\Delta t} \cdot \mathrm{d}\tau^{-T}_{\xi_j \Delta t} \cdot (\nabla_{\xi_i} \nabla_{\xi_j} L_1 + \nabla_{\xi_i} \nabla_{\xi_j} L_2) \cdot \mathrm{d}\tau^{-1}_{\xi_i \Delta t} \cdot \mathrm{Ad}_{\xi_i \Delta t} \\ & - \frac{\delta_{i,j}}{2\Delta t} (\nabla^T_{\xi_i} L_1 \otimes I + \nabla^T_{\xi_i} L_2 \otimes I) \cdot \partial_{\xi_i} (\mathrm{d}\tau^{-T}_{\xi \Delta t})^S \\ & \cdot \mathrm{d}\tau^{-1}_{\xi_i \Delta t} \cdot \mathrm{Ad}_{\xi_i \Delta t} \end{aligned}$$

$$[D_1 D_2 L_d(g_k, g_{k+1})]_{i,j} = [D_2 D_1 L_d(g_k, g_{k+1})]^T_{j,i}$$

$$\begin{aligned} & [D_2 D_2 L_d(g_k, g_{k+1})]_{i,j} \\ =& \frac{\Delta t}{2} \nabla_{E_i} \nabla_{E_j} L_2 + \frac{1}{2} \mathrm{Ad}^T_{\tau(\xi_j \Delta t)} \cdot \mathrm{d}\tau^{-T}_{\xi_j \Delta t} \cdot \nabla_{E_i} \nabla_{\xi_j} L_2 + \\ & \frac{1}{2} \nabla_{\xi_i} \nabla_{E_j} L_2 \cdot \mathrm{d}\tau^{-1}_{\xi_i \Delta t} \cdot \mathrm{Ad}_{\tau(\xi_i \Delta t)} + \frac{1}{2\Delta t} \mathrm{Ad}^T_{\tau(\xi_j \Delta t)} \cdot \\ & \mathrm{d}\tau^{-T}_{\xi_j \Delta t} \cdot (\nabla_{\xi_i} \nabla_{\xi_j} L_1 + \nabla_{\xi_i} \nabla_{\xi_j} L_2) \cdot d\tau^{-1}_{\xi_i \Delta t} \cdot \mathrm{Ad}_{\tau(\xi_i \Delta t)} + \\ & \frac{\delta_{i,j}}{2\Delta t} (\nabla^T_{\xi_i} L_1 \otimes I + \nabla^T_{\xi_i} L_2 \otimes I) \cdot \partial_{\xi_i} (\mathrm{Ad}^T_{\tau(\xi_i \Delta t)} \cdot d\tau^{-T}_{\xi_i \Delta t})^S \cdot \\ & \mathrm{d}\tau^{-1}_{\xi_i \Delta t} \cdot \mathrm{Ad}_{\tau(\xi \Delta t)}. \end{aligned}$$

Note if $i \neq j$, $[D_{(\cdot)} D_{(\cdot)}]_{i,j} = [D_{(\cdot)} D_{(\cdot)}]^T_{j,i}$; and if G_i is abelian, $[D_{(\cdot)} D_{(\cdot)}]_{i,i} = [D_{(\cdot)} D_{(\cdot)}]^T_{i,i}$.

D. Structured Linearization of Constrained Systems

For constrained systems, the Lie group variational integrator firstly solves Eqs. (10a) and (10b) to get g_{k+1} and λ_k, and then updates the discrete momentum p_{k+1} through Eq. (10c).

The structured linearization of constrained systems on Lie groups is quite similar to that in generalized coordinates [12]. Here we only take $\frac{\partial g_{k+1}}{\partial g_k}$ as an example and the other components can be obtained with the same approach.

The Lagrange multipliers only depend on g_k, p_k and u_k, so by taking variation on Eqs. (10a) and (10b), we can get

$$\frac{\partial g_{k+1}}{\partial g_k} = -M^{-1}_{k+1} [C_{g_k} - Dh^T(g_k) \frac{\partial \lambda_k}{\partial g_k}] \tag{17}$$

$$Dh(g_{k+1}) \frac{\partial g_{k+1}}{\partial g_k} = 0 \tag{18}$$

where $C_{g_k} = D_1 D_1 L_{k+1} + D_1 F^-_{k+1} - D^2 h^T(g_k) \lambda_k$.

Substitute Eq. (17) to Eq. (18):

$$Dh(g_{k+1}) \cdot M^{-1}_{k+1} \cdot Dh^T(g_k) \frac{\partial \lambda_k}{\partial g_k} - Dh(g_{k+1}) \cdot M^{-1}_{k+1} \cdot C_{g_k} = 0$$

If $Dh(g_{k+1}) \cdot M^{-1}_{k+1} \cdot Dh^T(g_k)$ is invertible, we have

$$\frac{\partial \lambda_k}{\partial g_k} = [Dh(g_{k+1}) M^{-1}_{k+1} Dh^T(g_k)]^{-1} Dh(g_{k+1}) M^{-1}_{k+1} C_{g_k} \tag{19}$$

and then substitute Eq. (19) back to Eq. (17), $\frac{\partial g_{k+1}}{\partial g_k}$ is well defined.

IV. Implementation

In this section, we implement the derived structure linearization on two examples: the asymmetric 3D pendulum [22]–[25] and the quadrotor with a suspended load [26]–[28], which synthesizes the current analysis and control tools. Both systems have been studied extensively by the robotic community and various controllers have been designed by exploring the system dynamics. Instead of suggesting new control methods, we apply LQR controllers constructed through structured linearization to these systems. The simulation results show these LQR controllers work very well and have a large basin of attraction. Especially for the asymmetric 3D pendulum, the LQR controller never fails in our tests.

In the simulation we choose the Cayley map rather than the more standard exponential map as the retraction. The Cayley map is a 3rd approximation of the exponential map, i.e. $\mathrm{cay}(\xi) = \exp(\xi) + O(\|\xi\|^3)$, and most importantly, it doesn't involve trigonometric functions and the derivatives are much easier to compute [6].

[3] ∇_{E_i} denotes derivatives of Lie algebra elements in $T_eG_i \subset T_eQ$ whereas ∇_g are derivatives of elements from the whole T_eQ.

A. LQR Control on Lie Groups

LQR control is one of the most commonly-used methods in trajectory tracking. For a discretized reference trajectory (g_{k_d}, p_{k_d}) on Lie groups, where $g_{k_d} \in Q$ and p_{k_d} is the discrete momentum (Eq. (9b)), the LQR problems can be formulated as seeking control inputs u_k^* to minimize the quadratic cost

$$V(x_0, \mu(\cdot)) = \sum_{k=0}^{N-1} [x_k^T Q_k x_k + \mu_k^T R_k \mu_k] + x_N^T Q_N x_N \quad (20)$$

subject to the linearized dynamics

$$x_{k+1} = A_k x_k + B_k \mu_k \quad (21)$$

where Q_k, R_k are positive-definite and symmetric, x_k and μ_k are perturbations from the desired trajectory

$$x_k = \begin{bmatrix} \log(g_{k_d}^{-1} g_k) \\ p_k - p_{k_d} \end{bmatrix} \text{ and } \mu_k = u_k - u_{k_d}.$$

Here the operation $\log(\cdot)$ is obtained from the logarithm operation on each G_i through the direct product representation.

Similar to vector spaces [4], LQR controllers on Lie groups are also obtained by solving the discrete Ricatti equation and the control law is [29]:

$$u_k^* = u_{k_d} - \mathcal{K}_k x_k$$

where $\mathcal{K}_k$ is found iteratively backwards in time:

$$\mathcal{K}_k = (R_k + B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k$$
$$P_k = \mathcal{K}_k^T R_k \mathcal{K}_k + Q_k + (A_k - B_k \mathcal{K}_k)^T P_{k+1} (A_k - B_k \mathcal{K}_k)$$
$$P_N = Q_N.$$

B. Example: The Forced Asymmetric 3D Pendulum

The 3D pendulum is a rigid body supported by a fixed pivot point having three rotation freedoms and acted upon by a uniform gravity force as well as some control and disturbance forces, whose configuration space is $SO(3)$. Literatures about dynamics and control of the 3D pendulum can be found in [22]–[25].

Some papers about the control of axially symmetric 3D pendulum have been published [24], in which case the model is reduced to a 2D spherical pendulum. However, the asymmetric 3D pendulum, whose three principal moments of inertia are distinct and center of mass is not at the pivot location, demonstrates much richer and much more complex dynamics, which is very hard to control. Though in [30] it is proved that the hanging and inverted equilibrium of the asymmetric 3D pendulum is asymptotically stable, there seems still a lack of robust controllers to do trajectory tracking. Here we employ LQR controllers to track trajectories of the asymmetric 3D pendulum, which performs very well.

[4]Essentially the perturbation component $\eta_k = \log(g_{k_d}^{-1} g_k)$ is pulled pack from the tangent space at g_k through g_k^{-1}. Hence, though all η_k are Lie algebra elements in $T_e Q$, they should have different geometric meanings if Q is non-abelian. However, this difference has no influence on the construction of the LQR controller.

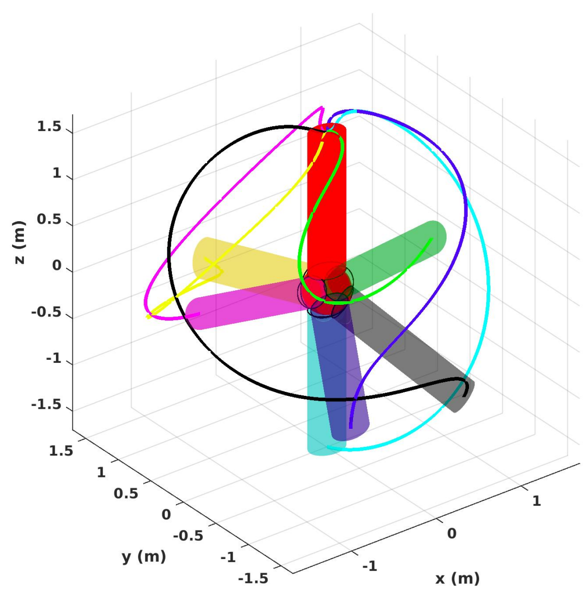


Fig. 1: Stabilize the 3D pendulum to the inverted equilibrium. The pendulum marked by red is at the inverted equilibrium while pendulums of the other colors denote different initial conditions. The solid lines are tip trajectories of the pendulum.

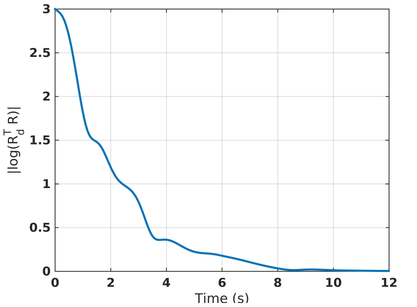
The dynamic equations used are from [23]. The simulation example is an axially asymmetric one with $J = \text{diag}[3, 7, 4]\text{kg} \cdot \text{m}^2$, $m = 1\text{kg}$ and $L = 1.5\text{m}$. The system has 3-dimensional control inputs that are torques exerted on the pendulum body frame. Through structured linearization we can get A_k, B_k in Eq. (21) and then construct the LQR controller with diagonal matrices for each cost matrix with an entry of 100 for the configuration variables and identity everywhere else.

The first test is to stabilize the pendulum to the inverted equilibrium (Fig. 1). The second test is to track the trajectory of an unforced pendulum with random initial position and angular velocity. Note that the reference angular velocity (Fig. 2(b)) is no longer periodical. We also construct a LQR controller through structured linearization in Euler angles with `trep` [31] as comparison, which works well with relatively small initial errors but fails if the perturbed initial condition is far away from the referenced one (Fig. 2(c)). From both examples, it can be seen clearly that the controller can stabilize the system even with large initial errors, as listed in Figs. 1 and 2.

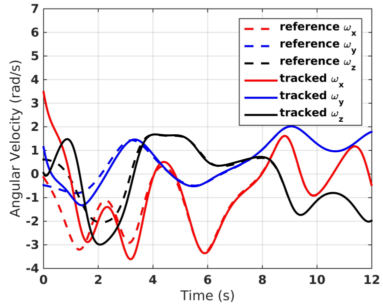
C. Example: Quadrotor with a Suspended Load

The quadrotor with a suspended load (Fig. 3) has been studied in [26]–[28]. This system is differentially flat with the quadrotor yaw angle and the load position as flat outputs. In [26], [27] a controller is designed to track the flat outputs, which leads to the tracking of all states.

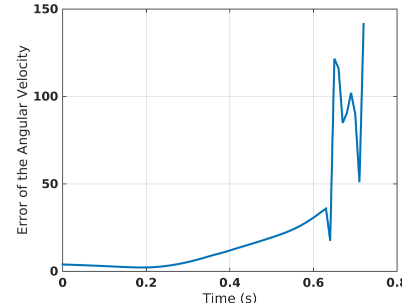
Here we model it as a constrained system whose configuration space is $(R, x_q, x_L) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, where (R, x_q) is the attitude and position of the quadrotor and



(a) Tracking performance of the pendulum attitude

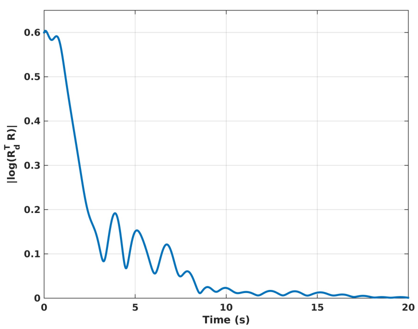


(b) Tacking performance of the angular velocity

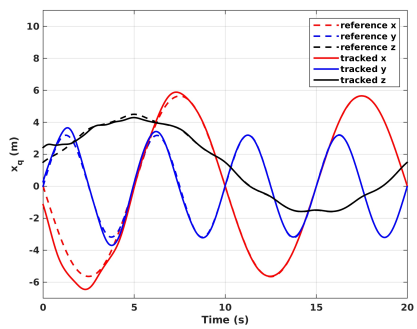


(c) Failture of an LQR controller constructed with XYZ Euler angles, the angular velocity error $\|\omega - \omega_d\|$ rises very quickly

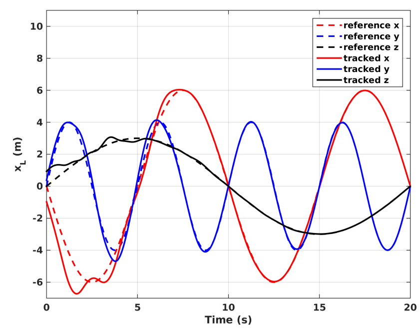
Fig. 2: Track a randomly generated trajectory of a unforced 3D pendulum. The reference initial condition is $\alpha_d = -0.30$, $\beta_d = -0.37$, $\gamma_d = 0.55$ and $\omega_d = [-0.14 \quad -0.47 \quad 0.62]^T$ rad/s while the perturbed initial condition is $\alpha = 0.30$, $\beta = 1.12$, $\gamma = -2.70$ and $\omega_0 = [3.48 \quad 1.14 \quad 0.05]^T$ rad/s, where α_d, β_d, γ_d and α, β, γ are roll, pitch, yaw of XYZ Euler angles.



(a) Tracking performance of the quadrotor attitude



(b) Tacking performance of the quadrotor position



(c) Tacking performance of the load position

Fig. 4: Trajectory tracking of the quadrotor with suspended load. The reference initial condition is $\alpha_d = 0$, $\beta_d = 0$, $\gamma_d = 0$, $x_{q_d} = [0 \quad 0 \quad 1.5]^T$ m, $x_{l_d} = [0 \quad 0 \quad 0]^T$ m while the perturbed initial condition is $\alpha = 0.23$, $\beta = -0.32$, $\gamma = 0.49$, $x_q = [-1.15 \quad 0.28 \quad 2.411]^T$ m, $x_l = [-0.98 \quad 0.38 \quad 0.93]^T$ m, where α_d, β_d, γ_d and α, β, γ are roll, pitch, yaw of XYZ Euler angles.

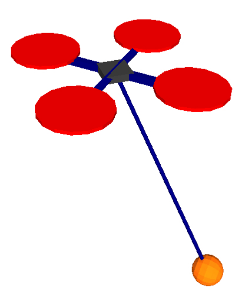


Fig. 3: A quadrotor with a suspended load.

x_L is the position of the suspended load [26], [27], with the constraint $\|x_q - x_L\| = L$ imposed. Here we adopt a simplified quadrotor model whose control inputs are the rotor angular velocities $\omega_i\,(i = 1, 2, 3, 4)$ and assume the cable is always in tension. Note by choosing angular velocity ω_i instead of the square of rotor angular velocities ω_i^2 as inputs, the LQR controller captures an important feature of quadrotor dynamics that downward forces can't be exerted on the quadrotor body, which is neglected in some papers on quadrotor control.

In our simulation, a LQR controller is constructed with diagonal matrices for each cost matrix with an entry of 50 for the configuration variables, 2 for control inputs and identity everywhere else. The reference flat outputs x_L and yaw angle γ are all trigeometric functions and then the system is linearized around the reference trajectory by structured linearization for constrained systems. The performance of the LQR controller is as Fig. 4, which successfully tracks all states of the system despite the relatively large initial errors as listed in the caption of Fig. 4.

V. Conclusion

In this paper we derive structured linearization for forced and constrained mechanical systems on Lie groups, assuming that a variational integrator is used to represent time evolution. We demonstrate that vector space methods can be directly applied to the control of mechanical systems on Lie groups, even when the Lie group is non-abelian. Lastly we use two motivated example systems - the asymmetric 3D pendulum and quadrotor with a suspended load to indicate the utility of these methods.

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